

ELEMENTARY GEOMETRIC LOCAL-GLOBAL PRINCIPLES FOR FIELDS

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ABSTRACT. We define and investigate a family of local-global principles for fields involving both orderings and p -valuations. This family contains the PAC, PRC and PpC fields and exhausts the class of pseudo classically closed fields. We show that the fields satisfying such a local-global principle form an elementary class, admit diophantine definitions of holomorphy domains, and their orderings satisfy the strong approximation property.

1. INTRODUCTION

1.1. Geometric local-global principles. The topic of this work is the study of *geometric local-global principles* for fields from a model theoretic point of view. Here, a field F is said to satisfy a geometric local-global principle for a class of F -varieties \mathcal{V} and a family \mathcal{F} of extensions of F if each $V \in \mathcal{V}$ has an F -rational point if and only if it has F' -rational points for all $F' \in \mathcal{F}$. For example, the classical Hasse-Minkowski theorem tells us that a quadric V over $F = \mathbb{Q}$ has a \mathbb{Q} -rational point if and only if it has rational points over each of the completions $\mathbb{R}, \mathbb{Q}_2, \mathbb{Q}_3, \dots$ of \mathbb{Q} . However, this does not hold for arbitrary \mathbb{Q} -varieties V . We are interested in fields F that satisfy a geometric local-global principle for *all* F -varieties.

A well-studied class of such fields consists of Prestel's *pseudo real closed* (PRC) fields, defined by the property that every F -variety that has a smooth rational point over every real closure of F has an F -rational point [Pre81], [Ers83], [Pre85] – a prominent example of a field with this property is the field \mathbb{Q}_{tr} of totally real algebraic numbers. Among other things, it was shown that the class of PRC fields is elementary in the language of rings. That is, the PRC property can be formulated in (possibly infinitely many) sentences of first-order logic. Similar work was done for the p -adic analogue, the PpC fields [Gro87], [HJ88], [Kün89b]. Examples of further modifications and generalizations are [Kün89a], [Ers92], and [Dar00].

1.2. Pseudo classically closed fields. The aim of this work is to give a common framework for several geometric local-global principles that came up in recent years. For example, let S be a finite set of absolute values on a number field K and let $K_{\text{tot},S}$ denote the maximal Galois extension of K contained in all of the completions $\hat{K}_{\mathfrak{p}}$, $\mathfrak{p} \in S$ – the field of *totally S -adic numbers*. It was proven that the field $K_{\text{tot},S}$, as well as certain subfields F of $K_{\text{tot},S}$ satisfy a geometric local-global principle – they are *pseudo- S closed* (PSC): A K -variety V that has smooth $\hat{K}_{\mathfrak{p}}$ -rational points for all $\mathfrak{p} \in S$ has F -rational points, [MB89], [GPR95], [Pop96], [JR98], [GJ02]. This notion of PSC fields has been defined and studied only for algebraic extensions of K .

Another class of interest consists of the pseudo classically closed (PCC) fields of [Pop03]. The class of PCC fields contains all PRC and all PpC fields, and the notion PCC is defined for arbitrary fields. Note however, that the class of PCC fields is not elementary.

In this work we define a family of local-global principles for fields of characteristic zero, all of which are elementary. Both PRC, PpC and PSC fields are special cases, and all PCC fields are covered.

1.3. Results. Let K be a number field, S a finite set of orderings and valuations on K , $\tau = (e, f) \in \mathbb{N}^2$ a pair of positive integers, and F an extension of K . For $\mathfrak{p} \in S$ we denote by $\mathcal{S}_{\mathfrak{p}}^{\tau}(F)$ the set of all orderings and p -valuations of F extending \mathfrak{p} , where in the case of p -valuations we demand in addition that the relative initial ramification index and residue degree are at most e resp. f . We say that F is $PS^{\tau}CC$ if F satisfies a geometric local-global principle for all F -varieties with respect to the family of real and p -adic closures of F at elements of $\bigcup_{\mathfrak{p} \in S} \mathcal{S}_{\mathfrak{p}}^{\tau}(F)$.

Note the following special cases¹:

- (1) $S = \emptyset$: F is $PS^{\tau}CC \Leftrightarrow F$ is PAC (see e.g. [FJ08, Chapter 11])
- (2) $K = \mathbb{Q}$, $S = \{\infty\}$: F is $PS^{\tau}CC \Leftrightarrow F$ is PRC
- (3) $K = \mathbb{Q}$, $S = \{p\}$, $\tau = (1, 1)$: F is $PS^{\tau}CC \Leftrightarrow F$ is PpC
- (4) $K = \mathbb{Q}$: F is $PS^{\tau}CC$ for some S and $\tau \Leftrightarrow F$ is PCC
- (5) $\tau = (1, 1)$, $F \subseteq K_{\text{tot}, S}$: F is $PS^{\tau}CC \Leftrightarrow F$ is PSC

In particular, our main results generalize the corresponding results for PRC and PpC fields:

Theorem 1.1. *The class of $PS^{\tau}CC$ fields is elementary in the language $\mathcal{L}_{\text{ring}}(K)$ of rings with constants from K .*

The most important ingredient in the proof is the following definability result:

Theorem 1.2. *If F is $PS^{\tau}CC$ and $\mathfrak{p} \in S$, then the holomorphy domain*

$$\bigcap_{\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^{\tau}(F)} \mathcal{O}_{\mathfrak{P}}$$

where $\mathcal{O}_{\mathfrak{P}}$ is the positive cone resp. valuation ring of \mathfrak{P} , is uniformly diophantine in F over K .

This means that this holomorphy domain is the projection of the zero set of a polynomial over K which is independent of F .

Prestel proved that the orderings of any PRC fields satisfy the so-called strong approximation property: Given an open-closed set of orderings one can find an element which is positive at all of those, and negative at all the other orderings. We show that this result extends to $PS^{\tau}CC$ fields:

Theorem 1.3. *If F is $PS^{\tau}CC$, then $\mathcal{S}_{\mathfrak{p}}^{\tau}(F)$ satisfies the strong approximation property for each $\mathfrak{p} \in S$.*

Extending the notion of totally real field extensions we call an extension E/F *totally S^{τ} -adic* if every element of $\mathcal{S}_{\mathfrak{p}}^{\tau}(F)$ extends to an element of $\mathcal{S}_{\mathfrak{p}}^{\tau}(E)$ of the same type (i.e. same residue field and same initial ramification, in the case of p -valuations). Combining Theorems 1.1, 1.2 and 1.3 we get the following corollary.

Corollary 1.4. *If F is $PS^{\tau}CC$ and $F \prec F^*$ is an elementary extension, then F^*/F is totally S^{τ} -adic.*

These results have immediate consequences for PCC fields:

Corollary 1.5. *Let F be a PCC field.*

¹Here, ∞ denotes the unique ordering and p denotes the p -adic valuation on \mathbb{Q} .

- (1) *The intersection over all p -valuation rings of F for any p , as well as the intersection over all positive cones of F , are existentially \emptyset -definable in F .*
- (2) *If $E \equiv F$, then E is PCC.*
- (3) *If $F \prec E$, then every ordering and every p -valuation of F extends to an ordering resp. p -valuation of E of the same type.*
- (4) *The space of orderings of F satisfies the strong approximation property.*

In fact, we prove everything in greater generality, without the assumption that K is a number field. The results of this work also answer a question posed by Darnière in [Dar01] and play a crucial role in the axiomatization and proof of decidability of $K_{\text{tot},S}$ and certain subfields of it in [Feh12].

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2. PRELIMINARIES AND NOTATION

2.1. Notation. Every ring and every semiring is commutative with 1. If R is a ring, we denote by R^\times the group of invertible elements of R . If K is a field, we denote by \bar{K} a fixed algebraic closure of K . The cardinality of a set X is denoted by $|X|$. By \cup we denote the disjoint union of sets. Varieties are geometrically irreducible and geometrically reduced. If V is a K -variety and $K \subseteq L$ a field extension we denote by $L(V)$ the function field of V over L .

2.2. Model Theory. For the basic notions of model theory see for example [Mar02]. The language of rings is $\mathcal{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$ where $+$ and \cdot are binary function symbols, $-$ is a unary function symbol, and 0 and 1 are constant symbols. If \mathcal{L} is a language containing $\mathcal{L}_{\text{ring}}$, K is an \mathcal{L} -structure, and C is a subset of K , we denote by $\mathcal{L}(C) = \mathcal{L} \cup \{c_x : x \in C\}$ the language \mathcal{L} augmented by constant symbols for the elements in C . If $\varphi(x_1, \dots, x_n)$ is an \mathcal{L} -formula in n free variables, and K is an \mathcal{L} -structure, we denote by $\varphi(K) = \{\mathbf{a} \in K^n : K \models \varphi(\mathbf{a})\}$ the subset defined by φ in K .

2.3. Real Closed Fields. We assume familiarity with the theory of ordered and real closed fields as presented in [Pre84], and only recall a few definitions and facts.

A **positive cone** of a field K is a semiring $P \subseteq K$ (i.e. $0, 1 \in P$, $P+P \subseteq P$, $P \cdot P \subseteq P$) such that $P \cup (-P) = K$ and $P \cap (-P) = \{0\}$. A field is **real closed** if it has an ordering but each proper algebraic extension has no ordering. A real closed field K has a unique ordering, given by the positive cone K^2 , [Pre84, 3.2]. A real closed field F is a **real closure** of an ordered field K if F is an algebraic extension of K and the unique ordering of F extends the ordering of K . Any ordered field K has a real closure, which is unique up to K -isomorphism, [Pre84, 3.10].

The language of ordered rings $\mathcal{L}_{\leq} = \mathcal{L}_{\text{ring}} \cup \{\leq\}$ is the language of rings augmented by a binary relation symbol \leq , which is interpreted as the ordering of an ordered field. The \mathcal{L}_{\leq} -theory of real closed ordered fields is complete and has effective quantifier elimination, [Mar02, 3.3.15, 3.3.16].

2.4. Valued Fields. We assume familiarity with the basics of valuation theory, see e.g. [EP05].

If $v : K \rightarrow \Gamma \cup \{\infty\}$ is a valuation on a field K with value group Γ we denote by \mathcal{O}_v the valuation ring, by \mathfrak{m}_v its maximal ideal, and by $\bar{K}_v = \mathcal{O}_v / \mathfrak{m}_v$ the residue field. We say that v is of **rank one** if its value group has no non-trivial proper convex subgroup, and **discrete** if its value group is discrete in the order topology. We normalize every discrete valuation such that \mathbb{Z} is a convex subgroup of the value group. We will use the following variant of Hensel's lemma, [EP05, 4.1.3(5)].

Lemma 2.1. *Let v be a Henselian valuation on K . If $f \in \mathcal{O}_v[X]$ and $a \in \mathcal{O}_v$ with $v(f(a)) > 2v(f'(a))$, then there exists $\alpha \in \mathcal{O}_v$ with $f(\alpha) = 0$ and $v(a - \alpha) > v(f'(a))$.*

The language of valued fields $\mathcal{L}_R = \mathcal{L}_{\text{ring}} \cup \{R\}$ is the language of rings augmented by a unary predicate symbol R , which is interpreted as the valuation ring of a valued field.

2.5. p -adically Closed Fields. We recall the notion of p -adically closed fields and quote some well known results from [PR84].

A valuation v on a field K of characteristic zero with residue field of characteristic $p > 0$ is a **p -valuation of p -rank $d \in \mathbb{N}$** if $\dim_{\mathbb{F}_p} \mathcal{O}_v / p\mathcal{O}_v = d$. The residue field \bar{K}_v of a p -valued field (K, v) is finite, and the value group $v(K^\times)$ is discrete and $v(p) \in \mathbb{Z}$. If $e = v(p)$ and $f = [\bar{K}_v : \mathbb{F}_p]$, then $d = ef$, [PR84, p. 15]. We call (p, e, f) the **type** of (K, v) .

A p -valued field is **p -adically closed** if it has no proper p -valued algebraic extension of the same p -rank. Every p -adically closed valued field (K, v) has a unique p -valuation, [PR84, 6.15]. A **p -adic closure** of a p -valued field (K, v) is an algebraic extension of (K, v) which is p -adically closed of the same p -rank as (K, v) . A p -valued field (K, v) is p -adically closed if and only if it is Henselian and the value group $v(K^\times)$ is a \mathbb{Z} -group, [PR84, 3.1]. Here, an ordered abelian group Γ is a **\mathbb{Z} -group** if it is discrete and $(\Gamma : n\Gamma) = n$ for each $n \in \mathbb{N}$. Any p -valued field (K, v) has a p -adic closure. A p -adic closure of (K, v) is unique up to K -isomorphism if and only if $v(K^\times)$ is a \mathbb{Z} -group, [PR84, 3.2].

3. CLASSICAL PRIMES

We start by introducing the notion of a classical prime. This notion generalizes the notion of a place of a number field and unifies considerations about orderings and p -valuations.

Definition 3.1. A **prime \mathfrak{p}** of a field K is either an equivalence class of valuations on K (\mathfrak{p} is a **non-archimedean** prime) or an ordering of K (\mathfrak{p} is an **archimedean** prime). If \mathfrak{p} is an equivalence class of valuations, let $v_{\mathfrak{p}}$ be a fixed valuation in the class \mathfrak{p} , let $p_{\mathfrak{p}} = \text{char}(\bar{K}_{\mathfrak{p}})$, the characteristic of the **residue field** $\bar{K}_{\mathfrak{p}} = \bar{K}_{v_{\mathfrak{p}}}$, and denote by

$$\mathcal{O}_{\mathfrak{p}} = \{x \in K : v_{\mathfrak{p}}(x) \geq 0\}$$

the corresponding valuation ring. If \mathfrak{p} is an ordering, denote \mathfrak{p} by $\leq_{\mathfrak{p}}$, let $p_{\mathfrak{p}} = \infty$, and denote by

$$\mathcal{O}_{\mathfrak{p}} = \{x \in K : x \geq_{\mathfrak{p}} 0\}$$

the corresponding positive cone. The **localization** $K_{\mathfrak{p}}$ of K with respect to \mathfrak{p} is a Henselization of $(K, v_{\mathfrak{p}})$ (if $p_{\mathfrak{p}} \neq \infty$) resp. a real closure of $(K, \leq_{\mathfrak{p}})$ (if $p_{\mathfrak{p}} = \infty$). It is unique up to K -isomorphism.

Remark 3.2. The reader may have noticed that our definition of primes does not include the classical so called ‘complex primes’, i.e. absolute values for which the corresponding completion is isomorphic to \mathbb{C} . The reason for this omission is that both for the PSC

property and for the definition of the fields $K_{\text{tot},S}$ we are interested in, the ‘complex primes’ in S can be disregarded.

Example 3.3. *The field \mathbb{Q} has one archimedean prime, which we denote by ∞ , and one non-archimedean prime for each prime number p , which we simply denote by p . Note that in our notation, \mathbb{Q}_p is now the field of p -adic algebraic numbers – we denote the field of p -adic numbers by $\hat{\mathbb{Q}}_p$.*

Definition 3.4. Let F/K be an extension of fields. A prime \mathfrak{P} of F **lies above** a prime \mathfrak{p} of K if $\mathcal{O}_{\mathfrak{P}} \cap K = \mathcal{O}_{\mathfrak{p}}$. We write this as $\mathfrak{P}|_K = \mathfrak{p}$. If \mathfrak{p} is a prime of K and $\sigma \in \text{Aut}(K)$ is an automorphism of K , then the **conjugate** $\sigma\mathfrak{p}$ of \mathfrak{p} is the unique prime of K with $\mathcal{O}_{\sigma\mathfrak{p}} = \sigma(\mathcal{O}_{\mathfrak{p}})$.

Definition 3.5. A **classical** prime \mathfrak{p} of K is either an equivalence class of p -valuations, for some prime number p , or an ordering of K . For a classical prime \mathfrak{p} of K , a **classical closure** of (K, \mathfrak{p}) is a p -adic closure of $(K, v_{\mathfrak{p}})$ resp. a real closure of $(K, \leq_{\mathfrak{p}})$. Let $\text{CC}(K, \mathfrak{p})$ denote the set of all classical closures of (K, \mathfrak{p}) contained in \tilde{K} . We say that (K, \mathfrak{p}) is **classically closed** if $K \in \text{CC}(K, \mathfrak{p})$, i.e. if K is p -adically closed resp. real closed. A prime \mathfrak{p} of K is **local** if it is classical and the value group of $v_{\mathfrak{p}}$ is isomorphic to \mathbb{Z} resp. the ordering $\leq_{\mathfrak{p}}$ is archimedean. A classical prime \mathfrak{p} of K is **quasi-local** if $K_{\mathfrak{p}} \in \text{CC}(K, \mathfrak{p})$, i.e. if the localization is a classical closure.

Remark 3.6. Note that this definition of local primes essentially coincides with the definition of local primes in [GJ02] and [HJP09a], and the ‘classical p -adic valuations and orderings’ in [HJP09b], except for the complex primes (cf. Remark 3.2). A non-archimedean classical prime is quasi-local if and only if its value group is a \mathbb{Z} -group, cf. Section 2.5. If \mathfrak{p} is quasi-local, then all $K' \in \text{CC}(K, \mathfrak{p})$ are K -conjugate. Each prime of a number field is local, and each local prime is quasi-local.

Definition 3.7. The **type** $\text{tp}(\mathfrak{p}) = (p_{\mathfrak{p}}, e_{\mathfrak{p}}, f_{\mathfrak{p}})$ of a classical prime \mathfrak{p} of K is the type (p, e, f) of the p -valuation $v_{\mathfrak{p}}$ if $p_{\mathfrak{p}} = p$, and $(\infty, 1, 1)$ if $p_{\mathfrak{p}} = \infty$. If \mathfrak{P} lies above \mathfrak{p} , then the **relative type** of \mathfrak{P} over \mathfrak{p} is $\text{tp}(\mathfrak{P}/\mathfrak{p}) = (e_{\mathfrak{P}}/e_{\mathfrak{p}}, f_{\mathfrak{P}}/f_{\mathfrak{p}}) \in \mathbb{N}^2$. We introduce a partial ordering on the set \mathbb{N}^2 of relative types by defining $(e, f) \leq (e', f')$ if $e \leq e'$ and $f|f'$, and a partial ordering on the set of types by defining $(p, e, f) \leq (p', e', f')$ if $p = p'$, $e \leq e'$ and $f|f'$. Since in a classically closed field (F, \mathfrak{P}) the prime \mathfrak{P} is unique, [HJP05, Prop. 7.2(c)], we write $\mathfrak{P}_F = \mathfrak{P}$ and $\text{tp}(F) = \text{tp}(\mathfrak{P}_F)$.

Definition 3.8. We say that a field F is **PFC** with respect to a family \mathcal{F} of algebraic extensions of F if every smooth F -variety V has an F -rational point, provided it has an F' -rational point for each $F' \in \mathcal{F}$, cf. [Jar91, §7].

If \mathcal{S} is a set of primes of F , then F is **pseudo- \mathcal{S} -closed with respect to localizations (PSCL)** if it is PFC with respect to the family

$$\mathcal{F} = \{F_{\mathfrak{P}} : \mathfrak{P} \in \mathcal{S}\}$$

of localizations. If \mathcal{S} is a set of classical primes of F , then F is **pseudo- \mathcal{S} -closed with respect to classical closures (PSCC)** if it is PFC with respect to the family

$$\mathcal{F} = \bigcup_{\mathfrak{p} \in \mathcal{S}} \text{CC}(F, \mathfrak{p})$$

of classical closures. If \mathcal{S} is a set of primes of F , then

$$R(\mathcal{S}) = \bigcap_{\mathfrak{p} \in \mathcal{S}} \mathcal{O}_{\mathfrak{p}}$$

is the **holomorphy domain**² of \mathcal{S} .

Remark 3.9. Since every classical closure is Henselian resp. real closed, PSCC implies PSCL . However, the converse does not hold.

4. PS^τCC , PS^τCL , AND PCC FIELDS

In this section we define the class of fields we are working with. For the rest of this work, we fix the following setting.

Setting 4.1.

- K is a fixed base field of characteristic 0.
- S is a finite set of local primes of K .
- $\tau \in \mathbb{N}^2$ is a relative type.
- F is an extension of K .

Definition 4.2. For $\mathfrak{p} \in S$ denote by $\mathcal{S}_{\mathfrak{p}}^\tau(F)$ the set of all classical primes \mathfrak{P} of F lying above \mathfrak{p} with $\text{tp}(\mathfrak{P}/\mathfrak{p}) \leq \tau$. Also, let

$$\begin{aligned}\mathcal{S}_S^\tau(F) &= \bigcup_{\mathfrak{p} \in S} \mathcal{S}_{\mathfrak{p}}^\tau(F), \\ R_{\mathfrak{p}}^\tau(F) &= R(\mathcal{S}_{\mathfrak{p}}^\tau(F)), \\ \text{CC}_{\mathfrak{p}}^\tau(F) &= \bigcup_{\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^\tau(F)} \text{CC}(F, \mathfrak{P}), \\ \text{CC}_S^\tau(F) &= \bigcup_{\mathfrak{p} \in S} \text{CC}_{\mathfrak{p}}^\tau(F).\end{aligned}$$

We say that F is **pseudo- \mathcal{S}^τ -closed with respect to localizations** (PS^τCL) resp. **pseudo- \mathcal{S}^τ -closed with respect to classical closures** (PS^τCC) if F is PSCL resp. PSCC with respect to $\mathcal{S} = \mathcal{S}_S^\tau(F)$.

Remark 4.3. Note that F is PS^τCC if and only if it is PFC with respect to the family $\mathcal{F} = \text{CC}_S^\tau(F)$. If F is PS^τCC , then F is PS^τCL , cf. Remark 3.9. In the case $\tau = (1, 1)$ we will drop τ in all notations, and write for example PSCC instead of PS^τCC .

Note that for $K = \mathbb{Q}$ and $|S| = 1$, our notion of PSCC fields coincides with the classical notions of PpC resp. PRC fields. For $K = \mathbb{Q}$ and S a finite set of prime numbers (cf. Example 3.3), the notion of PSCC fields coincides with the notion of PC_M fields of [Kün89a] and [Kün92]. For $K = \mathbb{Q}$ and $S = \emptyset$, a PSCC field is just a PAC field, cf. [FJ08, Chapter 11].

Note that there is a related notion of PSC fields in the literature (cf. the Introduction). However, in [JR98] and [GJ02] this property is defined only for algebraic extensions of K , and in [JR01], [Raz02] and [HJP09a] only for subextensions of $K_{\text{tot}, S}/K$. For subextensions of $K_{\text{tot}, S}/K$, the three notions PSC , PSCL , and PSCC coincide, but both the PSCL property and the PSCC property are defined for arbitrary extensions of K . The reason for our focus on the PSCC property is that, as we show, it is elementary.

We now briefly recall Pop's definition of a pseudo classically closed field and show how it fits into the picture.

Definition 4.4. Let $\text{CC}(F)$ denote the set of all classical closures of F with respect to arbitrary classical primes of F . A **classical field** is either \mathbb{R} or a finite extension of $\hat{\mathbb{Q}}_p$

²Note that if S contains archimedean primes, then $R(S)$ is only a semiring but not a ring.

for some p . If E is a classical field, let $\text{loc}^E(F)$ be the set of all algebraic extensions of F that are $\mathcal{L}_{\text{ring}}$ -elementarily equivalent to E . A field F is **PCC** if there exists a finite family of classical fields \mathcal{E} such that F is **PFC** for $\mathcal{F} = \bigcup_{E \in \mathcal{E}} \text{loc}^E(F)$.³

Lemma 4.5. *If F is **PFC** with respect to $\mathcal{F} = \bigcup_{E \in \mathcal{E}} \text{loc}^E(F)$ for a finite family of classical fields \mathcal{E} , then F is also **P \mathcal{F}_{\min} C**, where \mathcal{F}_{\min} is the set of minimal elements of \mathcal{F} . Moreover, $\mathcal{F}_{\min} = \text{CC}(F)$.*

Proof. See [Pop03, Theorem 2.3 and Corollary 2.11]. □

Proposition 4.6. *A field is **PCC** if and only if it is **PS $^\tau$ CC** for some finite set S of primes of $K = \mathbb{Q}$ and some $\tau \in \mathbb{N}^2$.*

Proof. Suppose F is **PFC** with respect to $\mathcal{F} = \bigcup_{E \in \mathcal{E}} \text{loc}^E(F)$ for a finite family of classical fields $\mathcal{E} = \{E_1, \dots, E_n\}$. If $\text{tp}(E_i) = (p_i, e_i, f_i)$, let $S = \{p_1, \dots, p_n\}$, $e = e_1 \cdots e_n$, $f = f_1 \cdots f_n$, and $\tau = (e, f)$. By Lemma 4.5 it follows that F is **P \mathcal{F}_{\min} C**, and that $\mathcal{F}_{\min} = \text{CC}(F)$. But then $\text{CC}(F) = \text{CC}_S^\tau(F)$, since if $F' \equiv E_i$, then F' is classically closed and $\text{tp}(F') = \text{tp}(E_i)$, [HJP05, Prop. 7.2(h)]. Thus, F is **PS $^\tau$ CC**.

Conversely, let F be **PS $^\tau$ CC**. Since every classically closed field is elementarily equivalent to a classical field, and only finitely many types occur among $\text{CC}_S^\tau(F)$, each of which is the type of only finitely many classical fields, [HJP05, Prop. 7.2(j),(k)], there exists a finite family of classical fields \mathcal{E} such that $\text{CC}_S^\tau(F) \subseteq \mathcal{F} := \bigcup_{E \in \mathcal{E}} \text{loc}^E(F)$. It follows that F is **PFC**, and hence **PCC**. □

Definition 4.7. We say that F is **S^τ -quasi-local** if every $\mathfrak{P} \in \mathcal{S}_S^\tau(F)$ is quasi-local (cf. Definition 3.5).

Lemma 4.8. *If F/K is algebraic, then F is **S^τ -quasi-local**.*

Proof. This follows from the assumption that S consists of local primes: If $\mathfrak{p} \in S$ is a p -valuation with value group \mathbb{Z} and $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^\tau(F)$, then the value group of \mathfrak{P} is discrete and contained in the divisible hull of \mathbb{Z} since F/K is algebraic, hence it is isomorphic to \mathbb{Z} itself. □

Proposition 4.9. *If F is **PS $^\tau$ CC**, then F is **S^τ -quasi-local**.*

Proof. By Proposition 4.6, F is **PFC** with respect to $\mathcal{F} = \bigcup_{E \in \mathcal{E}} \text{loc}^E(F)$ for a finite family of classical fields \mathcal{E} . By Lemma 4.5, F is **P \mathcal{F}_{\min} C**, and $\mathcal{F}_{\min} = \text{CC}(F)$, hence $\text{CC}_S^\tau(F) \subseteq \mathcal{F}_{\min}$. Thus the claim follows from [Pop03, Theorem 2.5 and Theorem 2.9]. □

5. DEFINING HOLOMORPHY DOMAINS

This section contains the technical first-order definition of the holomorphy domains. For a moment we forget about K and S and consider the following setting.

Setting 5.1.

- F is a field of characteristic zero.
- \mathcal{S} is a set of classical⁴ primes of F .
- \mathcal{S} is partitioned as $\mathcal{S} = \bigcup_{i=1}^n \mathcal{S}_i$.
- For each i , $(p'_i, e'_i, f'_i) \leq (p_i, e_i, f_i)$ are types such that $p_{\mathfrak{P}} = p_i$ and $f_{\mathfrak{P}} | f_i$ for each $\mathfrak{P} \in \mathcal{S}_i$.

³This definition coincides with the original one since the **PFC** property is preserved under enlarging \mathcal{F} .

⁴This condition can be weakened. For example, most results of this section apply also to valuations of residue characteristic zero.

- For each i , π_i is an element of F^\times that satisfies the following conditions:
 - (S1) If $\mathfrak{P} \in \mathcal{S}_i$ and $p_i \neq \infty$, then $v_{\mathfrak{P}}(\pi_i) > 0$ and $v_{\mathfrak{P}}(\pi_i) \leq e_i$.
 - (S2) If $\mathfrak{P} \in \mathcal{S} \setminus \mathcal{S}_i$ and $p_{\mathfrak{P}} \neq \infty$, then $v_{\mathfrak{P}}(\pi_i - 1) > 0$.
 - (S3) If $\mathfrak{P} \in \mathcal{S}_i$ and $p_i = \infty$, then $\pi_i <_{\mathfrak{P}} -1$.
 - (S4) If $\mathfrak{P} \in \mathcal{S} \setminus \mathcal{S}_i$ and $p_{\mathfrak{P}} = \infty$, then $\pi_i >_{\mathfrak{P}} 0$.

Definition 5.2. Let $\pi = \prod_{i=1}^n \pi_i$ and

$$\mathcal{S}'_i = \{\mathfrak{P} \in \mathcal{S}_i : v_{\mathfrak{P}}(\pi_i) \leq e'_i, f_{\mathfrak{P}}|f'_i\}$$

if $p_i \neq \infty$, and $\mathcal{S}'_i = \mathcal{S}_i$ if $p_i = \infty$.

Our first goal is to give a first-order definition of the holomorphy domain $R(\mathcal{S}'_i)$ in the case that F is PSCL. The case $n = m = 1$ of the following lemma can be found in [HP84].

Lemma 5.3. *Let $f \in F[X_1, \dots, X_n]$ and $g \in F[Y_1, \dots, Y_m]$ be non-constant polynomials, and let $c \in F^\times$. If g is square-free in $\tilde{F}[\mathbf{Y}]$, then*

$$h(\mathbf{X}, \mathbf{Y}) = f(\mathbf{X})g(\mathbf{Y}) + c \in F[\mathbf{X}, \mathbf{Y}]$$

is absolutely irreducible.

Proof. Without loss of generality assume that $F = \tilde{F}$. We prove the claim by induction on n .

First assume that $n = 1$. Let $r(\mathbf{Y})$ be any prime factor of $g(\mathbf{Y})$. Since g is square-free, $r|g$ but $r^2 \nmid g$. Write h as a polynomial in X_1 . Then r divides all coefficients of h except the constant one. Thus, by Eisenstein's criterion, h is irreducible in $F(\mathbf{Y})[\mathbf{X}]$. Since $c \neq 0$, it follows that h is irreducible in $F[\mathbf{X}, \mathbf{Y}]$.

Now assume that $n > 1$ and $f \notin F[X_1]$. Suppose that h decomposes as $h = h_1 h_2$ with $h_1, h_2 \in F[\mathbf{X}, \mathbf{Y}] \setminus F$. Since $c \neq 0$ we have $h_1, h_2 \notin F[X_1]$: Indeed, if, say, $h_1 \in F[X_1]$, then looking at the constant term of h with respect to \mathbf{Y} gives $h_1(X_1)|f(\mathbf{X})g(0) + c$, while each non-constant term gives $h_1(X_1)|f(\mathbf{X})$, a contradiction. Hence, there exists $x \in F$ such that $h_1(x, X_2, \dots, X_n, \mathbf{Y}) \notin F$, $h_2(x, X_2, \dots, X_n, \mathbf{Y}) \notin F$, and $f(x, X_2, \dots, X_n) \notin F$. Consequently, $f(x, X_2, \dots, X_n)g(\mathbf{Y}) + c$ decomposes in $F[X_2, \dots, X_n, \mathbf{Y}]$, contradicting the induction hypothesis. \square

Lemma 5.4. *Let $f \in F[X_1, \dots, X_n]$ be non-constant, and let $g \in F[Y]$ be non-constant and square-free in $\tilde{F}[Y]$ with $g(1) \neq 0$ and $g'(1) \neq 0$. Then the polynomial*

$$G(\mathbf{X}, Y) = g(Y)(1 + f(\mathbf{X})) - g(1) \in F[\mathbf{X}, Y]$$

is absolutely irreducible, and for every root \mathbf{x} of f , $(\mathbf{x}, 1)$ is a non-singular point on the hypersurface defined by G .

Proof. This follows from Lemma 5.3 and a direct computation. \square

Our formula defining $R(\mathcal{S}'_i)$ makes use of a polynomial of the form $G(\mathbf{X}, Y)$ in Lemma 5.4. More precisely, we let $f(\mathbf{X})$ depend on a parameter $a \in F$ such that $R(\mathcal{S}'_i)$ consists of all $a \in F$ for which $G(\mathbf{X}, Y)$ has a zero in F . We construct $f(\mathbf{X})$ as a product of several polynomials, each of which has a zero in a certain class of localizations of F , so that the hypersurface $G = 0$ has a smooth point in every localization. The basic idea for this approach appears in [Kün89a].

Lemma 5.5. *Under Setting 5.1, the polynomial*

$$A_i(X) = X^{2^{e_i}} - \pi_i$$

satisfies the following conditions:

- (A1) If $\mathfrak{P} \in \mathcal{S} \setminus \mathcal{S}_i$ and $p_{\mathfrak{P}} \neq 2$, then A_i has a zero in $F_{\mathfrak{P}}$.
- (A2) If $\mathfrak{P} \in \mathcal{S}_i$ and $p_i \neq \infty$, then for all $x \in F$, $v_{\mathfrak{P}}(A_i(x)) \leq e_i$.
- (A3) If $\mathfrak{P} \in \mathcal{S}_i$ and $p_i \neq \infty$, then $v_{\mathfrak{P}}(A_i(1)) = 0$.
- (A4) $A_i(X)$ is square-free in $\tilde{F}[X]$, and $A'_i(1) \neq 0$.
- (A5) If $\mathfrak{P} \in \mathcal{S}_i$ and $p_i = \infty$, then for all $x \in F$, $A_i(x) >_{\mathfrak{P}} 1$.

Proof. (A1): If $p_{\mathfrak{P}} \neq \infty$, then (A1) follows from (S2) and Hensel's lemma, otherwise it follows from (S4) and the fact that $F_{\mathfrak{P}}$ is real closed. (A2): The inequality $e_i < 2^{e_i}$ implies that $v_{\mathfrak{P}}(x^{2^{e_i}}) \neq v_{\mathfrak{P}}(\pi_i) \leq e_i$ by (S1). (A3) follows from (S1), (A4) from $\text{char}(F) = 0$, and (A5) from (S3). \square

Lemma 5.6. *Under Setting 5.1, the polynomial*

$$B_i(X) = X^{2^{e_i}} + \pi_i X + \pi$$

satisfies the following conditions:

- (B1) If $\mathfrak{P} \in \mathcal{S} \setminus \mathcal{S}_i$ and $p_{\mathfrak{P}} = 2$, then B_i has a zero in $F_{\mathfrak{P}}$.
- (B2) If $\mathfrak{P} \in \mathcal{S}_i$ and $p_i \neq \infty$, then for all $x \in F$, $v_{\mathfrak{P}}(B_i(x)) \leq e_i$.

Proof. (B1) follows from Hensel's lemma. (B2): The inequality $e_i < 2^{e_i}$ implies that $v_{\mathfrak{P}}(B_i(x)) = v_{\mathfrak{P}}(\pi) \leq e_i$ if $v_{\mathfrak{P}}(x) > 0$, and $v_{\mathfrak{P}}(B_i(x)) = v_{\mathfrak{P}}(x^{2^{e_i}}) \leq 0$ if $v_{\mathfrak{P}}(x) \leq 0$. \square

Lemma 5.7. *Under Setting 5.1, if $p_i \neq \infty$, then for every $a \in F$ the polynomial*

$$D_{i,a}(X) = a\pi_i X^{2^{e_i}} - X + a$$

satisfies the following conditions:

- (D1) If $\mathfrak{P} \in \mathcal{S}_i$ and $v_{\mathfrak{P}}(a) \geq 0$, then $D_{i,a}$ has a zero in $F_{\mathfrak{P}}$.
- (D2) If $\mathfrak{P} \in \mathcal{S}_i$ and $v_{\mathfrak{P}}(a) < 0$, then $v_{\mathfrak{P}}(D_{i,a}(x)) \leq v_{\mathfrak{P}}(a)$ for all $x \in F$. Thus, if $v_{\mathfrak{P}}(D_{i,a}(x)) \geq 0$ for some $x \in F$, then $v_{\mathfrak{P}}(a) \geq 0$.

Proof. (D1) follows from Hensel's lemma and (S1). (D2): If $v_{\mathfrak{P}}(x) \geq 0$, then $v_{\mathfrak{P}}(a\pi_i x^{2^{e_i}}) > v_{\mathfrak{P}}(a)$ and $v_{\mathfrak{P}}(x) > v_{\mathfrak{P}}(a)$, so $v_{\mathfrak{P}}(D_{i,a}(x)) = v_{\mathfrak{P}}(a)$. If $v_{\mathfrak{P}}(x) < 0$, then the inequality $2^{e_i} \geq e_i + 1$ implies that $v_{\mathfrak{P}}(a\pi_i x^{2^{e_i}}) < 2^{e_i}v_{\mathfrak{P}}(x) + v_{\mathfrak{P}}(\pi_i) \leq -e_i + v_{\mathfrak{P}}(x) + v_{\mathfrak{P}}(\pi_i) \leq v_{\mathfrak{P}}(x)$ and $v_{\mathfrak{P}}(a\pi_i x^{2^{e_i}}) = v_{\mathfrak{P}}(a) + v_{\mathfrak{P}}(\pi_i) + 2^{e_i}v_{\mathfrak{P}}(x) \leq v_{\mathfrak{P}}(a) + e_i - 2^{e_i} < v_{\mathfrak{P}}(a)$, so $v_{\mathfrak{P}}(D_{i,a}(x)) = v_{\mathfrak{P}}(a\pi_i x^{2^{e_i}}) < v_{\mathfrak{P}}(a)$. \square

Lemma 5.8. *Under Setting 5.1 and $p_i \neq \infty$, let $d \leq e_i$. Then the polynomial*

$$R_{i,d}(X, Y) = (X^{2d} + \pi_i^2)Y^{2^{e_i}} - X^d Y + \pi_i^{-1}X^{2d} + \pi_i$$

satisfies the following condition:

- (R1) If $\mathfrak{P} \in \mathcal{S}_i$ with $d|v_{\mathfrak{P}}(\pi_i)$, then $R_{i,d}$ has a zero in $F_{\mathfrak{P}}$.
- (R2) If $\mathfrak{P} \in \mathcal{S}_i$ with $d \nmid v_{\mathfrak{P}}(\pi_i)$, then $v_{\mathfrak{P}}(R_{i,d}(x, y)) \leq e_i$ for all $x, y \in F$.

Proof. First note that with $\gamma(X) = \pi_i^{-1}X^d + \pi_i X^{-d}$ we have $R_{i,d}(X, Y) = X^d D_{i,\gamma(X)}(Y)$. Furthermore, note that for $x \in F^\times$ and $\mathfrak{P} \in \mathcal{S}_i$, $v_{\mathfrak{P}}(\gamma(x)) \geq 0$ if and only if $dv_{\mathfrak{P}}(x) = v_{\mathfrak{P}}(\pi_i)$.

(R1): There exists $x \in F^\times$ with $dv_{\mathfrak{P}}(x) = v_{\mathfrak{P}}(\pi_i)$, i.e. $v_{\mathfrak{P}}(\gamma(x)) \geq 0$. Therefore, by (D1), $D_{i,\gamma(x)}(Y)$ has a zero $y \in F_{\mathfrak{P}}$, so (x, y) is a zero of $R_{i,d}$.

(R2): Since $v_{\mathfrak{P}}(\gamma(x)) < 0$ for all $x \in F^\times$, we have that $v_{\mathfrak{P}}(D_{i,\gamma(x)}(y)) \leq v_{\mathfrak{P}}(\gamma(x)) < 0$ for all $x \in F^\times$, $y \in F$ by (D2). Assume that there are $x, y \in F$ with $v_{\mathfrak{P}}(R_{i,d}(x, y)) > e_i$. Then $x \neq 0$, since $R_{i,d}(0, y) = \pi_i^2(y^{2^{e_i}} + \pi_i^{-1})$, and thus $v_{\mathfrak{P}}(R_{i,d}(0, y)) \leq v_{\mathfrak{P}}(\pi_i) \leq e_i$. It follows that $v_{\mathfrak{P}}(D_{i,\gamma(x)}(y)) > v_{\mathfrak{P}}(x^{-d}) + v_{\mathfrak{P}}(\pi_i) \geq v_{\mathfrak{P}}(\gamma(x))$, a contradiction. \square

Lemma 5.9. *Under Setting 5.1 and $p_i \neq \infty$, let $d|f_i$. Let*

$$I_{i,d}(X) = \Phi_{p_i^d-1}(X) \in \mathbb{Z}[X]$$

be the $(p_i^d - 1)$ -th cyclotomic polynomial. Then $I_{i,d}(X)$ satisfies the following conditions.

- (I1) *If $\mathfrak{P} \in \mathcal{S}_i$ with $d|f_{\mathfrak{P}}$, then $I_{i,d}$ has a zero in $F_{\mathfrak{P}}$.*
- (I2) *If $\mathfrak{P} \in \mathcal{S}_i$ with $d \nmid f_{\mathfrak{P}}$, then $v_{\mathfrak{P}}(I_{i,d}(x)) \leq 0$ for all $x \in F$.*

Proof. Note that $I_{i,d}$ has a zero in the finite field $\bar{F}_{\mathfrak{P}}$ if and only if $\mathbb{F}_{p_i^d} \subseteq \bar{F}_{\mathfrak{P}}$, [Bou88, V §11 Lemma 3], which is the case if and only if $d|f_{\mathfrak{P}}$. (I1) follows from Hensel's lemma. (I2) follows immediately since $I_{i,d}$ is monic. \square

Lemma 5.10. *Under Setting 5.1 and $p_i \neq \infty$, the polynomial*

$$N_i(X, Y) = \prod_{d \leq e_i, d \nmid e'_i} R_{i,d}(X, Y) \cdot \prod_{d|f_i, d \nmid f'_i} I_{i,d}(X)$$

satisfies the following conditions:

- (N1) *If $\mathfrak{P} \in \mathcal{S}_i \setminus \mathcal{S}'_i$, then N_i has a zero in $F_{\mathfrak{P}}$.*
- (N2) *If $\mathfrak{P} \in \mathcal{S}'_i$, then $v_{\mathfrak{P}}(N_i(x, y)) \leq e_i^2$ for all $x, y \in F$.*

Proof. (N1): If $v_{\mathfrak{P}}(\pi_i) \not\leq e'_i$, then $R_{i,v_{\mathfrak{P}}(\pi_i)}$ has a zero in $F_{\mathfrak{P}}$ by (R1). If $f_{\mathfrak{P}} \nmid f'_i$, then $I_{i,f_{\mathfrak{P}}}$ has a zero in $F_{\mathfrak{P}}$ by (I1).

(N2): Since $v_{\mathfrak{P}}(\pi_i) \leq e'_i$ and $f_{\mathfrak{P}}|f'_i$, it follows that for all $d \leq e_i$ with $d \not\leq e'_i$, we have that $d \nmid v_{\mathfrak{P}}(\pi_i)$, and for $d|f_i$ with $d \nmid f'_i$ we have $d \nmid f_{\mathfrak{P}}$. Therefore, by (R2) and (I2), $v_{\mathfrak{P}}(N_i(x, y)) \leq (e_i - e'_i)e_i \leq e_i^2$ for all $x, y \in F$. \square

Lemma 5.11. *Under Setting 5.1 and $p_i \neq \infty$, let $a \in F$. If A_i satisfies (A1)-(A4), B_i satisfies (B1)-(B2), $D_{i,a}$ satisfies (D1)-(D2), and N_i satisfies (N1)-(N2), then the polynomial*

$$G_{i,a}(X, Y, Z) = A_i(Z)(1 + \pi_i^{-4e_i - e_i^2} A_i(X)B_i(X)D_{i,a}(X)N_i(X, Y)) - A_i(1)$$

satisfies the following conditions:

- (1) *If $G_{i,a}$ has a zero in F , then $v_{\mathfrak{P}}(a) \geq 0$ for all $\mathfrak{P} \in \mathcal{S}'_i$.*
- (2) *If F is PSCL and $v_{\mathfrak{P}}(a) \geq 0$ for all $\mathfrak{P} \in \mathcal{S}'_i$, then $G_{i,a}$ has a zero in F .*

Proof. Let $x, y, z \in F$ with $G_{i,a}(x, y, z) = 0$ and let $\mathfrak{P} \in \mathcal{S}'_i$. Then

$$\begin{aligned} v_{\mathfrak{P}}(1 + \pi_i^{-4e_i - e_i^2} A_i(x)B_i(x)D_{i,a}(x)N_i(x, y)) &= \\ &= v_{\mathfrak{P}}(A_i(1)) - v_{\mathfrak{P}}(A_i(z)) \geq -e_i \end{aligned}$$

by (A2) and (A3). Thus,

$$v_{\mathfrak{P}}(\pi_i^{-4e_i - e_i^2} A_i(x)B_i(x)D_{i,a}(x)N_i(x, y)) \geq -e_i,$$

so $v_{\mathfrak{P}}(D_{i,a}(x)) \geq 0$ by (S1), (A2), and (B2), and (N2). Therefore, $v_{\mathfrak{P}}(a) \geq 0$ by (D2).

Now assume that F is PSCL and $v_{\mathfrak{P}}(a) \geq 0$ for all $\mathfrak{P} \in \mathcal{S}'_i$. If $A_i(1) = 0$, then $G_{i,a}(0, 0, 1) = 0$. Hence, assume without loss of generality that $A_i(1) \neq 0$. Let $\mathfrak{P} \in \mathcal{S}$. We claim that $A_i(X)B_i(X)D_{i,a}(X)N_i(X, Y)$ has a zero in $F_{\mathfrak{P}}$. If $\mathfrak{P} \in \mathcal{S} \setminus \mathcal{S}_i$ and $p_{\mathfrak{P}} \neq 2$, this follows from (A1). If $\mathfrak{P} \in \mathcal{S} \setminus \mathcal{S}_i$ and $p_{\mathfrak{P}} = 2$, this follows from (B1). If $\mathfrak{P} \in \mathcal{S}_i \setminus \mathcal{S}'_i$, this follows from (N1). If $\mathfrak{P} \in \mathcal{S}'_i$, this follows from (D1). Therefore, by Lemma 5.4 and (A4), $G_{i,a}$ is absolutely irreducible and has a simple zero in $F_{\mathfrak{P}}$ for all $\mathfrak{P} \in \mathcal{S}$. Since F is PSCL, $G_{i,a}$ has a zero in F . \square

This almost concludes the proof of the definability of $R(\mathcal{S}'_i)$ for $p_i \neq \infty$. We now turn to the case $p_i = \infty$.

Lemma 5.12. Under Setting 5.1, if $p_i = \infty$, then the polynomial

$$C(X) = X^2 + X + 2$$

satisfies the following conditions:

- (C1) If $\mathfrak{P} \in \mathcal{S} \setminus \mathcal{S}_i$ and $p_{\mathfrak{P}} = 2$, then C has a zero in $F_{\mathfrak{P}}$.
- (C2) If $\mathfrak{P} \in \mathcal{S}_i$, then $C(x) >_{\mathfrak{P}} 1$ for every $x \in F$.

Proof. (C1) follows from Hensel's lemma. (C2) is clear. \square

Lemma 5.13. Under Setting 5.1, if $p_i = \infty$, then for every $a \in F$, the polynomial

$$E_a(X) = X^2 - a$$

satisfies the following conditions:

- (E1) If $\mathfrak{P} \in \mathcal{S}_i$ and $a \geq_{\mathfrak{P}} 0$, then E_a has a zero in $F_{\mathfrak{P}}$.
- (E2) If $\mathfrak{P} \in \mathcal{S}_i$, $x, \epsilon \in F$, and $E_a(x) \leq_{\mathfrak{P}} \epsilon$, then $a \geq_{\mathfrak{P}} -\epsilon$.

Proof. (E1) holds since $F_{\mathfrak{P}}$ is real closed. (E2) is obvious. \square

Lemma 5.14. Under Setting 5.1, if $p_i = \infty$, then for every $u \in F^\times$, the polynomial

$$H_u(X) = X^2 + u^2$$

satisfies the following conditions:

- (H1) If $\mathfrak{P} \in \mathcal{S}_i$, then for all $x \in F$, $H_u(x) \geq_{\mathfrak{P}} u^2$.
- (H2) If $\mathfrak{P} \in \mathcal{S}_i$, then $H_u(1) = 1 + u^2 >_{\mathfrak{P}} 0$.
- (H3) $H_u(X)$ is square-free in $\tilde{F}[X]$, and $H'_u(1) \neq 0$.

Proof. All claims are easily verified. \square

Lemma 5.15. Under Setting 5.1 and $p_i = \infty$, let $a \in F$ and $u \in F^\times$. If A_i satisfies (A1) and (A5), C satisfies (C1)-(C2), E_a satisfies (E1)-(E2), and H_u satisfies (H1)-(H3), then the polynomial

$$G_{i,a,u}(X, Y) = H_u(Y)(1 + A_i(X)C(X)E_a(X)) - H_u(1)$$

satisfies the following conditions:

- (1) If $G_{i,a,u}$ has a zero in F , then $a \geq_{\mathfrak{P}} -u^{-2}$ for all $\mathfrak{P} \in \mathcal{S}_i$.
- (2) If F is PSCL and $a \geq_{\mathfrak{P}} 0$ for all $\mathfrak{P} \in \mathcal{S}_i$, then $G_{i,a,u}$ has a zero in F .

Proof. Let $x, y \in F$ such that $G_{i,a,u}(x, y) = 0$ and let $\mathfrak{P} \in \mathcal{S}_i$. Then

$$1 + A_i(x)C(x)E_a(x) = \frac{H_u(1)}{H_u(y)} \leq_{\mathfrak{P}} \frac{1 + u^2}{u^2} = 1 + u^{-2}$$

by (H1), (H2). Thus, $E_a(x) \leq_{\mathfrak{P}} u^{-2}$ by (A5) and (C2). Therefore, $a \geq_{\mathfrak{P}} -u^{-2}$ by (E2).

Now assume that F is PSCL and $a \geq_{\mathfrak{P}} 0$ for all $\mathfrak{P} \in \mathcal{S}_i$. If $H_u(1) = 0$, then $G_{i,a,u}(0, 1) = 0$. Hence, assume without loss of generality that $H_u(1) \neq 0$. Let $\mathfrak{P} \in \mathcal{S}$. We claim that $A_i(X)C(X)E_a(X)$ has a zero in $F_{\mathfrak{P}}$. If $\mathfrak{P} \in \mathcal{S} \setminus \mathcal{S}_i$ and $p_{\mathfrak{P}} \neq 2$, this follows from (A1). If $p_{\mathfrak{P}} = 2$, it follows from (C1). If $\mathfrak{P} \in \mathcal{S}_i$, it follows from (E1). Therefore, by Lemma 5.4, (H3), and the assumption that F is PSCL, it follows that $G_{i,a,u}$ has a zero in F . \square

For the following proposition, let $A_i, B_i, C, D_{i,a}, E_a, H_u, N_i$ be the concrete polynomials defined above.

Proposition 5.16. *Under Setting 5.1, for $p_i \neq \infty$ let $\varphi_i(a)$ be the $\mathcal{L}_{\text{ring}}(\pi_1, \dots, \pi_n)$ -formula*

$$(\exists x, y, z)(A_i(z)(1 + \pi_i^{-4e_i - e_i^2} A_i(x) B_i(x) D_{i,a}(x) N_i(x, y)) - A_i(1) = 0),$$

and for $p_i = \infty$ let $\varphi_i(a)$ be the $\mathcal{L}_{\text{ring}}(\pi_i)$ -formula

$$(\exists u \neq 0)(\exists x, y)(a(H_u(y)(1 + A_i(x)C(x)E_{a-u^{-2}}(x)) - H_u(1)) = 0).$$

Then the following holds for the subset $\varphi_i(F) \subseteq F$ defined by φ_i :

- (1) $\varphi_i(F) \subseteq R(\mathcal{S}'_i)$.
- (2) If F is PSCL, then $\varphi_i(F) = R(\mathcal{S}'_i)$.

Proof. For $p_i \neq \infty$, this follows directly from Lemma 5.11. For $p_i = \infty$, proceed as follows: If $a = 0$, then $a \in \varphi_i(F)$ and $a \in R(\mathcal{S}_i)$. If $a \in \varphi_i(F) \setminus \{0\}$, then Lemma 5.15(1) implies that for some $u \in F^\times$, $a - u^{-2} \geq_{\mathfrak{P}} -u^{-2}$ for all $\mathfrak{P} \in \mathcal{S}_i$, so $a \in R(\mathcal{S}_i)$. If F is PSCL and $a \in R(\mathcal{S}_i) \setminus \{0\}$, then a simple calculation shows that with $u = a^{-1}(a+1) \in F$, also $a - u^{-2} \in R(\mathcal{S}_i)$. Hence, by Lemma 5.15(2), $\varphi_i(a)$ is satisfied in F . \square

6. HOLOMORPHY DOMAINS IN PSCL FIELDS

Now we apply the general construction of the previous section to the fields we are interested in. We continue to work in Setting 4.1 and, for the rest of this paper, make the following additional assumptions:

- **For $\mathfrak{p} \in S$ with $p_{\mathfrak{p}} \neq \infty$, fix $\pi_{\mathfrak{p}} \in K$ with $v_{\mathfrak{p}}(\pi_{\mathfrak{p}}) = 1$.**
- **For $\mathfrak{p} \in S$ with $p_{\mathfrak{p}} = \infty$, let $\pi_{\mathfrak{p}} = -1$.**

Lemma 6.1. *Let $\tau' \leq \tau$ be a relative type and write $S = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$, $\tau = (e, f)$, and $\tau' = (e', f')$. Let $\mathcal{S}_i = \mathcal{S}_{\mathfrak{p}_i}^{\tau'}(F)$, $\mathcal{S} = \bigcup_{i=1}^n \mathcal{S}_i$, $p_i = p_{\mathfrak{p}_i}$, $e_i = e$, $f_i = f|_{\mathfrak{p}_i}$, $p'_i = p_{\mathfrak{p}_i}$, $e'_i = e'$, $f'_i = f'|_{\mathfrak{p}_i}$. Then there exist $\pi_1, \dots, \pi_n \in K$ such that the conditions of Setting 5.1 are satisfied.*

Proof. The existence of π_i follows from the weak approximation theorem applied to the finite set S , see e.g. [EP05, 1.1.3]. \square

Proposition 6.2. *Let $\mathfrak{p} \in S$ and $\tau' \leq \tau$ a relative type. There exists an existential $\mathcal{L}_{\text{ring}}(K)$ -formula $\theta_{R,\mathfrak{p}}^{\tau'}(z)$ that satisfies the following:*

- (1) $\theta_{R,\mathfrak{p}}^{\tau'}(F) \subseteq R_{\mathfrak{p}}^{\tau'}(F)$.
- (2) If F is $PS^{\tau}\text{CL}$, then $\theta_{R,\mathfrak{p}}^{\tau'}(F) = R_{\mathfrak{p}}^{\tau'}(F)$.

Proof. Apply Lemma 6.1 and assume $\mathfrak{p} = \mathfrak{p}_i$. Then the corresponding formula φ_i of Proposition 5.16 satisfies the claim. \square

This also proves Theorem 1.2 of the introduction. Indeed, if $p_i \neq \infty$, then the formula $\theta_{R,\mathfrak{p}}^{\tau}(z)$ is already diophantine (and independent of F). In the case $p_i = \infty$, the formula $\theta_{R,\mathfrak{p}}^{\tau}(z)$ is of the form

$$(\exists u \neq 0)(\exists x, y)(f(z, u, x, y) = 0)$$

with $f \in K[Z, U, X, Y]$ independent of F , which for a $PS^{\tau}\text{CC}$ field is equivalent to the diophantine formula

$$(\exists u, v, x, y)(f(z, u, x, y)^2 + (uv - 1)^2 = 0)$$

(note that if F is not real, then already $\theta_{R,\mathfrak{p}}^{\tau}(F) = F$).

Definition 6.3. Let $\mathfrak{p} \in S$ and $\tau' = (e', f') \leq \tau$. If $p_{\mathfrak{p}} \neq \infty$, let $q = p^{f'f_{\mathfrak{p}}}$, and define the **\mathfrak{p} -adic Kochen operator** over K of type τ' by

$$\gamma_{\mathfrak{p}}^{\tau'}(x) = \frac{1}{\pi_{\mathfrak{p}}} \cdot ((x^q - x) - (x^q - x)^{-1})^{-e'}$$

if this expression is well defined, and $\gamma_{\mathfrak{p}}^{\tau'}(x) = 0$ otherwise. Define the **\mathfrak{p} -adic Kochen ring** over K of type τ' of F by

$$\Gamma_{\mathfrak{p}}^{\tau'}(F) = \left\{ \frac{b}{1 + \pi_{\mathfrak{p}}c} : b, c \in \mathcal{O}_{\mathfrak{p}}[\gamma_{\mathfrak{p}}^{\tau'}(F)], 1 + \pi_{\mathfrak{p}}c \neq 0 \right\}.$$

If $p_{\mathfrak{p}} = \infty$, let $\gamma_{\mathfrak{p}}^{\tau'}(x) = \gamma(x) = x^2$ and $\Gamma_{\mathfrak{p}}^{\tau'}(F) = \mathcal{O}_{\mathfrak{p}}[\gamma(F)]$, the semiring generated by $\gamma(F)$ over $\mathcal{O}_{\mathfrak{p}}$.

Lemma 6.4. Let $\mathfrak{p} \in S$ and $\tau' \leq \tau$. Then $\mathcal{S}_{\mathfrak{p}}^{\tau'}(F) \neq \emptyset$ if and only if $\pi_{\mathfrak{p}}^{-1} \notin \Gamma_{\mathfrak{p}}^{\tau'}(F)$. In that case, if $p_{\mathfrak{p}} \neq \infty$, then $R_{\mathfrak{p}}^{\tau'}(F)$ is the integral closure of $\Gamma_{\mathfrak{p}}^{\tau'}(F)$; if $p_{\mathfrak{p}} = \infty$, then $R_{\mathfrak{p}}^{\tau'}(F) = \Gamma_{\mathfrak{p}}^{\tau'}(F)$.

Proof. For the case $p_{\mathfrak{p}} \neq \infty$ see [PR84, 6.4, 6.8, 6.9]. For the case $p_{\mathfrak{p}} = \infty$ note that if $-1 \notin \Gamma_{\mathfrak{p}}^{\tau'}(F)$, then $\Gamma_{\mathfrak{p}}^{\tau'}(F)$ is a pre-positive cone, so $\Gamma_{\mathfrak{p}}^{\tau'}(F) = R_{\mathfrak{p}}^{\tau'}(F)$, see [Pre84, 1.6]. \square

Definition 6.5. For $\mathfrak{p} \in S$ and $\tau' \leq \tau$, let $T_{R,\mathfrak{p}}^{\tau'}$ be the $\mathcal{L}_{\text{ring}}(K)$ -theory consisting of the following sentences.

- (1) A recursive set of sentences stating that $\theta_{R,\mathfrak{p}}^{\tau'}$ defines an integrally closed ring (if $p_{\mathfrak{p}} \neq \infty$) resp. a semiring (if $p_{\mathfrak{p}} = \infty$).
- (2) For every $a \in \mathcal{O}_{\mathfrak{p}}$ the sentence

$$\theta_{R,\mathfrak{p}}^{\tau'}(a).$$

- (3) The sentence

$$(\forall x)(\theta_{R,\mathfrak{p}}^{\tau'}(\gamma_{\mathfrak{p}}^{\tau'}(x))).$$

- (4) If $p_{\mathfrak{p}} \neq \infty$, the sentence

$$(\forall x)(\theta_{R,\mathfrak{p}}^{\tau'}(x) \wedge 1 + \pi_{\mathfrak{p}}x \neq 0 \rightarrow \theta_{R,\mathfrak{p}}^{\tau'}((1 + \pi_{\mathfrak{p}}x)^{-1})).$$

- (5) The sentence

$$\theta_{R,\mathfrak{p}}^{\tau'}(\pi_{\mathfrak{p}}^{-1}) \rightarrow (\forall x)(\theta_{R,\mathfrak{p}}^{\tau'}(x)).$$

Proposition 6.6. Let $\mathfrak{p} \in S$ and $\tau' \leq \tau$. Then F satisfies $T_{R,\mathfrak{p}}^{\tau'}$ if and only if the formula $\theta_{R,\mathfrak{p}}^{\tau'}$ defines the holomorphy domain $R_{\mathfrak{p}}^{\tau'}(F)$ in F .

Proof. This follows from Proposition 6.2(1), Lemma 6.4 and the definition of $\Gamma_{\mathfrak{p}}^{\tau'}(F)$. \square

7. QUANTIFICATION OVER CLASSICAL PRIMES

In this section we translate first-order statements concerning the classical primes of F to statements about F and the corresponding holomorphy domains.

Lemma 7.1. Let $\mathfrak{p} \in S$ with $p_{\mathfrak{p}} \neq \infty$, and $\tau' \leq \tau = (e, f)$. For $a \in F$ let

$$H_{\mathfrak{p}}^{\tau'}(a) = \{\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^{\tau'}(F) : a \in \mathcal{O}_{\mathfrak{P}}\}.$$

Then the following holds:

- (1) If $a, b \in F$, then $H_{\mathfrak{p}}^{\tau'}(a) \cap H_{\mathfrak{p}}^{\tau'}(b) = H_{\mathfrak{p}}^{\tau'}(a^{2^e} + \pi_{\mathfrak{p}}b^{2^e})$.
- (2) If $a \in F^{\times}$, then $\mathcal{S}_{\mathfrak{p}}^{\tau'}(F) \setminus H_{\mathfrak{p}}^{\tau'}(a) = H_{\mathfrak{p}}^{\tau'}((\pi_{\mathfrak{p}}a^{2^e})^{-1})$.

- (3) If $P(Z_1, \dots, Z_n)$ is a boolean polynomial⁵, then there exists a rational function $r(\mathbf{X}) \in \mathbb{Q}(\pi_{\mathfrak{p}})(X_1, \dots, X_n)$ independent of F such that for all $a_1, \dots, a_n \in F$,

$$P(H_{\mathfrak{p}}^{\tau'}(a_1), \dots, H_{\mathfrak{p}}^{\tau'}(a_n)) = H_{\mathfrak{p}}^{\tau'}(r(a_1, \dots, a_n)). \quad (7.1)$$

Proof. (1): Let $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$. If $v_{\mathfrak{P}}(a) \geq 0$ and $v_{\mathfrak{P}}(b) \geq 0$, then $v_{\mathfrak{P}}(a^{2^e} + \pi_{\mathfrak{p}}b^{2^e}) \geq 0$. If $v_{\mathfrak{P}}(a) < 0$ or $v_{\mathfrak{P}}(b) < 0$, then $v_{\mathfrak{P}}(a^{2^e} + \pi_{\mathfrak{p}}b^{2^e}) = \min\{v_{\mathfrak{P}}(a^{2^e}), v_{\mathfrak{P}}(\pi_{\mathfrak{p}}b^{2^e})\} < 0$, since $0 < v_{\mathfrak{P}}(\pi_{\mathfrak{p}}) \leq e' \leq e < 2^e$.

(2): Let $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$. If $v_{\mathfrak{P}}(a) \geq 0$, then $v_{\mathfrak{P}}(\pi_{\mathfrak{p}}a^{2^e}) \geq v_{\mathfrak{P}}(\pi_{\mathfrak{p}}) > 0$, so $v_{\mathfrak{P}}((\pi_{\mathfrak{p}}a^{2^e})^{-1}) < 0$. If $v_{\mathfrak{P}}(a) < 0$, then $v_{\mathfrak{P}}(a^{2^e}) \leq -2^e < -e \leq -v_{\mathfrak{P}}(\pi_{\mathfrak{p}})$, so $v_{\mathfrak{P}}((\pi_{\mathfrak{p}}a^{2^e})^{-1}) \geq 0$.

(3): If $\mathcal{S}_{\mathfrak{p}}^{\tau'}(F) = \emptyset$, then every $r(\mathbf{X})$ satisfies (7.1). Thus, assume that $\mathcal{S}_{\mathfrak{p}}^{\tau'}(F) \neq \emptyset$ and hence $a^{2^e} + \pi_{\mathfrak{p}} \neq 0$ for every $a \in F$. By (1), $H_{\mathfrak{p}}^{\tau'}(a) = H_{\mathfrak{p}}^{\tau'}(a) \cap H_{\mathfrak{p}}^{\tau'}(1) = H_{\mathfrak{p}}^{\tau'}(a^{2^e} + \pi_{\mathfrak{p}})$. Hence, the set of boolean polynomials $P(\mathbf{Z})$ for which there exists a rational function $r(\mathbf{X}) \in \mathbb{Q}(\pi_{\mathfrak{p}})(X_1, \dots, X_n)$ such that $r(\mathbf{a}) \notin \{0, \infty\}$ and (7.1) hold for all $a_1, \dots, a_n \in F$ contains Z_1, \dots, Z_n . By (1), it is closed under intersections. By (2), it is closed under complements. Hence, it contains all boolean polynomials. \square

Remark 7.2. In what comes, the predicate symbol R of the language \mathcal{L}_R will be used in two different ways. It will interpret either a valuation ring resp. positive cone $\mathcal{O}_{\mathfrak{P}}$, or a holomorphy domain $R_{\mathfrak{p}}^{\tau'}(F)$. We write $(F, \mathcal{O}_{\mathfrak{P}})$ and $(F, R_{\mathfrak{p}}^{\tau'}(F))$, respectively, for the corresponding structures.

Note that formally we work in the language $\mathcal{L}_{\text{ring}}$ of rings, i.e. there is no function \cdot^{-1} in our language. However, it is common to use this function in first-order formulas when working in fields, knowing that it can always be eliminated by introducing either an existential or a universal quantifier.

The following proposition makes explicit some ideas from [Pre81, p. 154] and [Gro87, proof of Theorem 4.01].

Proposition 7.3. *Let $\mathfrak{p} \in S$ and $\tau' \leq \tau$.*

- (1) *There exists a recursive map $\varphi(\mathbf{x}) \mapsto \varphi_{\mathfrak{p}, \exists}^{\tau'}(\mathbf{x})$ from existential \mathcal{L}_R -formulas to $\mathcal{L}_R(\pi_{\mathfrak{p}})$ -formulas such that for every extension F/K and elements $a_1, \dots, a_n \in F$ the following statements are equivalent:*
 - (1a) *There exists $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$ such that $(F, \mathcal{O}_{\mathfrak{P}}) \models \varphi(\mathbf{a})$.*
 - (1b) $(F, R_{\mathfrak{p}}^{\tau'}(F)) \models \varphi_{\mathfrak{p}, \exists}^{\tau'}(\mathbf{a})$.
- (2) *There exists a recursive map $\varphi(\mathbf{x}) \mapsto \varphi_{\mathfrak{p}, \forall}^{\tau'}(\mathbf{x})$ from universal \mathcal{L}_R -formulas to $\mathcal{L}_R(\pi_{\mathfrak{p}})$ -formulas such that for every extension F/K and elements $a_1, \dots, a_n \in F$ the following statements are equivalent:*
 - (2a) $(F, \mathcal{O}_{\mathfrak{P}}) \models \varphi(\mathbf{a})$ for all $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$.
 - (2b) $(F, R_{\mathfrak{p}}^{\tau'}(F)) \models \varphi_{\mathfrak{p}, \forall}^{\tau'}(\mathbf{a})$.

Proof. First of all, note that we can get (2) from (1) via $\varphi_{\mathfrak{p}, \forall}^{\tau'} := \neg(\neg\varphi)_{\mathfrak{p}, \exists}^{\tau'}$. Thus, it suffices to prove (1).

PART A1: CASE $p_{\mathfrak{p}} \neq \infty$. First assume that $\varphi(\mathbf{x})$ is of the simple form

$$\bigwedge_i (f_i(\mathbf{x}) \in R) \wedge \bigwedge_i (g_i(\mathbf{x}) \notin R),$$

⁵i.e. a term in the language of boolean algebras, cf. [FJ08, Chapter 7.6]

where $f_i, g_i \in \mathbb{Z}[\mathbf{X}]$ for all i . Let

$$\mathcal{H}(\mathbf{a}) = \bigcap_i H_p^{\tau'}(f_i(\mathbf{a})) \cap \bigcap_i (\mathcal{S}_p^{\tau'}(F) \setminus H_p^{\tau'}(g_i(\mathbf{a}))).$$

By Lemma 7.1(3) there exists a rational function $r \in \mathbb{Q}(\pi_p)(\mathbf{X})$ independent of F and \mathbf{a} such that $H_p^{\tau'}(r(\mathbf{a})) = \mathcal{S}_p^{\tau'}(F) \setminus \mathcal{H}(\mathbf{a})$. Then (1a) holds if and only if $\mathcal{H}(\mathbf{a}) \neq \emptyset$, that is, $H_p^{\tau'}(r(\mathbf{a})) \neq \mathcal{S}_p^{\tau'}(F)$, which is equivalent to $r(\mathbf{a}) \notin R_p^{\tau'}(F)$. Thus, if we let $\varphi_{p,\exists}^{\tau'}(\mathbf{x})$ be the formula $\neg(r(\mathbf{x}) \in R)$, then the claim follows.

PART A2: CONCLUSION OF THE PROOF FOR $p_p \neq \infty$. Now assume that $\varphi(\mathbf{x})$ is an arbitrary existential \mathcal{L}_R -formula in prenex disjunctive normal form, i.e. $\varphi(\mathbf{x})$ is of the form

$$(\exists y_1, \dots, y_m) \bigvee_j [\bigwedge_i (f_{ij}(\mathbf{x}, \mathbf{y}) \in R) \wedge \bigwedge_i (g_{ij}(\mathbf{x}, \mathbf{y}) \notin R) \wedge \bigwedge_i (h_{ij}(\mathbf{x}, \mathbf{y}) = 0) \wedge \bigwedge_i (k_{ij}(\mathbf{x}, \mathbf{y}) \neq 0)],$$

where $f_{ij}, g_{ij}, h_{ij}, k_{ij} \in \mathbb{Z}[\mathbf{X}, \mathbf{Y}]$. Let $\varphi_j(\mathbf{x}, \mathbf{y})$ be the formula

$$\bigwedge_i (f_{ij}(\mathbf{x}, \mathbf{y}) \in R) \wedge \bigwedge_i (g_{ij}(\mathbf{x}, \mathbf{y}) \notin R).$$

Then φ_j is of the special form considered in PART A1. Let $\varphi_{p,\exists}^{\tau'}(\mathbf{x})$ be the formula

$$(\exists y_1, \dots, y_m) \bigvee_j [(\varphi_j)_{p,\exists}^{\tau'}(\mathbf{x}, \mathbf{y}) \wedge \bigwedge_i (h_{ij}(\mathbf{x}, \mathbf{y}) = 0) \wedge \bigwedge_i (k_{ij}(\mathbf{x}, \mathbf{y}) \neq 0)].$$

Then $\varphi_{p,\exists}^{\tau'}$ satisfies the claim.

PART B1: CASE $p_p = \infty$. First assume that $\varphi(\mathbf{x})$ is of the form

$$\bigwedge_i (f_i(\mathbf{x}) \in R)$$

where $f_1, \dots, f_m \in \mathbb{Z}[\mathbf{X}]$. Assume that (1a) holds. Then there exists an ordering $\mathfrak{P} \in \mathcal{S}_p^{\tau'}(F)$ with $f_1(\mathbf{a}) \geq_{\mathfrak{P}} 0, \dots, f_m(\mathbf{a}) \geq_{\mathfrak{P}} 0$. Hence, $R_p^{\tau'}(F)[f_1(\mathbf{a}), \dots, f_m(\mathbf{a})]$, the semiring generated by $f_1(\mathbf{a}), \dots, f_m(\mathbf{a})$ over $R_p^{\tau'}(F)$, is contained in $\mathcal{O}_{\mathfrak{P}}$. In particular,

$$R_p^{\tau'}(F)[f_1(\mathbf{a}), \dots, f_m(\mathbf{a})] \cap (-R_p^{\tau'}(F)) = \{0\}, \quad (7.2)$$

so if $\varphi_{p,\exists}^{\tau'}(\mathbf{x})$ is the formula

$$\begin{aligned} (\forall s_1, \dots, s_r \in R) \quad & \left(- \sum_{j=1}^r s_j f_1(\mathbf{x})^{k_{j,1}} \dots f_m(\mathbf{x})^{k_{j,m}} \in R \right. \\ & \left. \rightarrow \sum_{j=1}^r s_j f_1(\mathbf{x})^{k_{j,1}} \dots f_m(\mathbf{x})^{k_{j,m}} = 0 \right), \end{aligned}$$

where $r = 2^m$, and $(k_{j,1}, \dots, k_{j,m})$ ranges over $\{0, 1\}^m$, then (1a) implies (1b).

Conversely, suppose that (1b) holds. Then, since $F^2 \subseteq R_p^{\tau'}(F)$, $\varphi_{p,\exists}^{\tau'}(\mathbf{a})$ holds in F even if $(k_{j,1}, \dots, k_{j,m})$ ranges over any finite subset of $(\mathbb{Z}_{\geq 0})^m$. Hence, (7.2) holds. Thus, $R_p^{\tau'}(F)[f_1(\mathbf{a}), \dots, f_m(\mathbf{a})]$ is a pre-positive cone, and hence there exists an ordering $\mathfrak{P} \in \mathcal{S}_p^{\tau'}(F)$ with $f_1(\mathbf{a}) \geq_{\mathfrak{P}} 0, \dots, f_m(\mathbf{a}) \geq_{\mathfrak{P}} 0$, [Pre84, 1.6]. That is, (1a) holds.

PART B2: CONCLUSION OF THE PROOF FOR $p_p = \infty$. Now assume that $\varphi(\mathbf{x})$ is an arbitrary existential \mathcal{L}_R -formula in prenex disjunctive normal form. Replace $x \notin R$ by $(-x \in R) \wedge (x \neq 0)$ and conclude the proof as in PART A2. \square

8. QUANTIFICATION OVER CLASSICAL CLOSURES

We use the quantification over classical primes of the previous section to quantify over classical closures.

We want to make use of the following purely model theoretic lemma, which we prove due to lack of a reference. Let $T_0 \subseteq T$ be theories in a language \mathcal{L} . Write $(M_0, M) \models (T_0, T)$ to indicate that M is a model of T , and M_0 is a substructure of M and a model of T_0 . Let $\Delta(M_0)$ denote the quantifier-free diagram of M_0 in the language $\mathcal{L}(M_0)$.

Lemma 8.1. *If $(M_0, M) \models (T_0, T)$ implies that $T \cup \Delta(M_0)$ is complete, then there exists a map $\varphi(\mathbf{x}) \mapsto \varphi^0(\mathbf{x})$ from \mathcal{L} -formulas to universal \mathcal{L} -formulas such that for every \mathcal{L} -formula $\varphi(\mathbf{x})$, the following holds:*

(1) $T \models \forall \mathbf{x}(\varphi(\mathbf{x}) \leftrightarrow \varphi^0(\mathbf{x}))$.

(2) If $(M_0, M) \models (T_0, T)$ and $\mathbf{a} \in M_0^r$, then $M \models \varphi^0(\mathbf{a})$ if and only if $M_0 \models \varphi^0(\mathbf{a})$.

If both T and T_0 are recursively enumerable, then the map $\varphi(\mathbf{x}) \mapsto \varphi^0(\mathbf{x})$ is recursive.

Proof (Itay Kaplan). Let

$$\Gamma = \{\alpha(\mathbf{x}) : \alpha \text{ universal } \mathcal{L}\text{-formula, and if } (M_0, M) \models (T_0, T) \\ \text{and } \mathbf{a} \in M_0^r, \text{ then } M \models \alpha(\mathbf{a}) \text{ if and only if } M_0 \models \alpha(\mathbf{a})\}$$

and

$$\Sigma = \{\alpha(\mathbf{x}) : \alpha \text{ universal } \mathcal{L}\text{-formula, } T \models \varphi \rightarrow \alpha\}.$$

Since $T_0 \subseteq T$, the assumption implies that T is model complete. Thus, we can assume without loss of generality that φ is universal, [Mar02, 3.4.12(d)]. If $A = \{\alpha_1, \dots, \alpha_n\} \subseteq \Sigma$ is a finite set and $\beta_A := \varphi \wedge \alpha_1 \wedge \dots \wedge \alpha_n$, then $T \models \varphi \leftrightarrow \beta_A$. Hence, if $\beta_A \in \Gamma$, then $\varphi^0 := \beta_A$ satisfies (1) and (2). Suppose that this does not happen, that is, $\beta_A \notin \Gamma$ for every finite set $A \subseteq \Sigma$. Since β_A is universal, this means that there exists $(M_0, M) \models (T_0, T)$ and $\mathbf{a} \in M_0^r$ with $M_0 \models \beta_A(\mathbf{a})$ and $M \models \neg \beta_A(\mathbf{a})$. Since $T \models \varphi \leftrightarrow \beta_A$, we get that $M \models \neg \varphi(\mathbf{a})$.

Let $\mathcal{L}^0 = \mathcal{L} \cup \{P, \mathbf{c}\}$, where P is a unary predicate symbol and $\mathbf{c} = (c_1, \dots, c_r)$ are constant symbols. Let the \mathcal{L}^0 -theory T^0 consist of the theory T and the statement that P defines a substructure that contains \mathbf{c} and is a model of T_0 . Let T^1 consists of T^0 , the sentence $\neg \varphi(\mathbf{c})$, and for every $\alpha \in \Sigma$ the statement that $\alpha(\mathbf{c})$ holds in the substructure defined by P .

By the above assumption, every finite subset of T^1 is consistent. Therefore the compactness theorem implies that T^1 has a model. That is, there exists $(M_0, M) \models (T_0, T)$ and $\mathbf{a} \in M_0^r$ such that $M \models \neg \varphi(\mathbf{a})$ and $M_0 \models \alpha(\mathbf{a})$ for every $\alpha \in \Sigma$.

Since by assumption $T \cup \Delta(M_0)$ is a complete $\mathcal{L}(M_0)$ -theory, $T \cup \Delta(M_0) \models \neg \varphi(\mathbf{a})$. Thus there exists $\psi(\mathbf{x}, \mathbf{y})$ and $\mathbf{b} \in M_0^s$ such that $\psi(\mathbf{a}, \mathbf{b}) \in \Delta(M_0)$ and $T \models \forall \mathbf{x} \forall \mathbf{y}(\psi(\mathbf{x}, \mathbf{y}) \rightarrow \neg \varphi(\mathbf{x}))$. Therefore, $(\forall \mathbf{y})(\neg \psi(\mathbf{a}, \mathbf{y})) \in \Sigma$. By construction of M_0 , this implies that $M_0 \models (\forall \mathbf{y})(\neg \psi(\mathbf{a}, \mathbf{y}))$, contradicting $\psi(\mathbf{a}, \mathbf{b}) \in \Delta(M_0)$. This contradiction shows that $\beta_A \in \Gamma$ for some A , as desired.

If both T_0 and T are recursively enumerable, then so is T^0 . Since a universal \mathcal{L} -formula β is in Γ if and only if $T^0 \vdash \beta(\mathbf{c}) \leftrightarrow \beta^P(\mathbf{c})$, where β^P is β with all quantifiers restricted to P , Γ is recursively enumerable. Thus one can recursively determine a universal \mathcal{L} -formula $\beta \in \Gamma$ with $T \vdash \varphi \leftrightarrow \beta$. Therefore, the map $\varphi \mapsto \varphi^0$ can be chosen recursive. \square

Note that the assumption of Lemma 8.1 is satisfied in particular if T is the **model completion** of T_0 , cf. [Mar02, 3.4.14]. Also, the cases $T_0 = \emptyset$ and $T_0 = T$ of Lemma 8.1 are well-known characterizations of quantifier elimination resp. model completeness.

Proposition 8.2. *For every type $\tau_1 = (p, e_1, f_1)$ there exists a recursive map $\varphi(\mathbf{x}) \mapsto \bar{\varphi}^{\tau_1}(\mathbf{x})$ from \mathcal{L}_R -formulas to universal \mathcal{L}_R -formulas with the following properties:*

- (1) *For every classically closed field (F', \mathfrak{P}) with $\text{tp}(\mathfrak{P}) = \tau_1$,*

$$(F', \mathcal{O}_{\mathfrak{P}}) \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \bar{\varphi}^{\tau_1}(\mathbf{x})).$$
- (2) *If \mathfrak{P} is a quasi-local prime of a field F with $\text{tp}(\mathfrak{P}) = \tau_1$ and $\mathbf{a} \in F^r$, then $(F, \mathcal{O}_{\mathfrak{P}}) \models \bar{\varphi}^{\tau_1}(\mathbf{a})$ if and only if $(F_{\mathfrak{P}}, \mathcal{O}_{F_{\mathfrak{P}}}) \models \bar{\varphi}^{\tau_1}(\mathbf{a})$.*
- (3) *If \mathfrak{P} is a prime of a field F with $\text{tp}(\mathfrak{P}) \leq \tau_1$ but $\text{tp}(\mathfrak{P}) \neq \tau_1$, and $\mathbf{a} \in F^r$, then $(F, \mathcal{O}_{\mathfrak{P}}) \models \bar{\varphi}^{\tau_1}(\mathbf{a})$.*

Proof. For $p = \infty$, this follows directly from quantifier elimination for real closed fields.

For $p \neq \infty$, apply Lemma 8.1 with T the theory of p -adically closed fields of type τ_1 and T_0 the theory of p -valued fields of type τ_1 with value group a \mathbb{Z} -group. The assumptions of Lemma 8.1 are satisfied by [PR84, 3.2, 3.4, 5.1] (in fact, this shows that T is the model completion of T_0). Therefore, if we let $\bar{\varphi}^{\tau_1}(\mathbf{x})$ be the formula $\varphi^0(\mathbf{x})$ of Lemma 8.1, then (1) and (2) are satisfied. In order to satisfy also (3), let ψ be the existential \mathcal{L}_R -sentence

$$(\exists x \in R)(x^{-1} \notin R \wedge px^{-e_1} \in R) \wedge (\exists x \in R)(\Phi_{p^{f_1-1}}(x)^{-1} \notin R)$$

where $\Phi_{p^{f_1-1}}$ is the $(p^{f_1} - 1)$ -th cyclotomic polynomial. Note that $(F, \mathcal{O}_{\mathfrak{P}}) \models \psi$ if and only if $\text{tp}(\mathfrak{P}) \geq \tau_1$. Thus, if we let $\bar{\varphi}^{\tau_1}(\mathbf{x})$ be the universal \mathcal{L}_R -formula $\psi \rightarrow \varphi^0(\mathbf{x})$, then (1)-(3) are satisfied.

Since the theories in question are axiomatized by recursive sets of sentences, and hence are recursively enumerable, the map $\varphi(\mathbf{x}) \mapsto \varphi^0(\mathbf{x})$ is recursive by Lemma 8.1. Therefore, also the map $\varphi(\mathbf{x}) \mapsto \bar{\varphi}^{\tau_1}(\mathbf{x})$ is recursive. \square

Lemma 8.3. *Let $\mathfrak{p} \in S$ and $\tau' \leq \tau$. There exists a recursive map $\varphi(\mathbf{x}) \mapsto \hat{\varphi}_{\mathfrak{p}, \mathbb{V}, R}^{\tau'}(\mathbf{x})$ from $\mathcal{L}_{\text{ring}}$ -formulas to $\mathcal{L}_R(K)$ -formulas such that for every extension F/K and elements $a_1, \dots, a_m \in F$ the following holds:*

- (1) *If $F' \models \varphi(\mathbf{a})$ holds for all $F' \in \text{CC}_{\mathfrak{p}}^{\tau'}(F)$ with $\text{tp}(\mathfrak{P}_{F'}/\mathfrak{p}) = \tau'$, then $(F, R_{\mathfrak{p}}^{\tau'}(F)) \models \hat{\varphi}_{\mathfrak{p}, \mathbb{V}, R}^{\tau'}(\mathbf{a})$.*
- (2) *If $(F, R_{\mathfrak{p}}^{\tau'}(F)) \models \hat{\varphi}_{\mathfrak{p}, \mathbb{V}, R}^{\tau'}(\mathbf{a})$ and $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$ with $\text{tp}(\mathfrak{P}/\mathfrak{p}) = \tau'$ is quasi-local, then $F_{\mathfrak{P}} \models \varphi(\mathbf{a})$.*

Proof. Write $\tau' = (e', f')$ and let $\tau_1 = (p_{\mathfrak{p}}, e'_{\mathfrak{p}}, f'_{\mathfrak{p}})$. Let $\psi(\mathbf{x})$ be the formula $\bar{\varphi}^{\tau_1}(\mathbf{x})$ of Proposition 8.2 and let $\hat{\varphi}_{\mathfrak{p}, \mathbb{V}, R}^{\tau'}(\mathbf{x})$ be the formula $\psi_{\mathfrak{p}, \mathbb{V}}^{\tau'}(\mathbf{x})$ that Proposition 7.3 attaches to $\psi(\mathbf{x})$. Then $\hat{\varphi}_{\mathfrak{p}, \mathbb{V}, R}^{\tau'}(\mathbf{x})$ satisfies the claim. \square

Proposition 8.4. *Let $\mathfrak{p} \in S$. There exists a recursive map $\varphi(\mathbf{x}) \mapsto \hat{\varphi}_{\mathfrak{p}, \mathbb{V}}^{\tau}(\mathbf{x})$ from $\mathcal{L}_{\text{ring}}$ -formulas to $\mathcal{L}_{\text{ring}}(K)$ -formulas such that for every extension F/K that satisfies $T_{R, \mathfrak{p}}^{\tau'}$ for all $\tau' \leq \tau$, and for all elements $a_1, \dots, a_m \in F$ the following holds:*

- (1) *If $F' \models \varphi(\mathbf{a})$ for all $F' \in \text{CC}_{\mathfrak{p}}^{\tau}(F)$, then $F \models \hat{\varphi}_{\mathfrak{p}, \mathbb{V}}^{\tau}(\mathbf{a})$.*
- (2) *If $F \models \hat{\varphi}_{\mathfrak{p}, \mathbb{V}}^{\tau}(\mathbf{a})$ and $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^{\tau}(F)$ is quasi-local, then $F_{\mathfrak{P}} \models \varphi(\mathbf{a})$.*

Proof. For $\tau' \leq \tau$ let $\psi^{\tau'}(\mathbf{x})$ be the formula $\hat{\varphi}_{\mathfrak{p}, \mathbb{V}, R}^{\tau'}(\mathbf{x})$ of Lemma 8.3 with all occurrences of $x \in R$ replaced by the formula $\theta_{R, \mathfrak{p}}^{\tau'}(x)$ of Proposition 6.2. Let $\hat{\varphi}_{\mathfrak{p}, \mathbb{V}}^{\tau}(\mathbf{x})$ be the formula $\bigwedge_{\tau' \leq \tau} \psi^{\tau'}$. Then $\hat{\varphi}_{\mathfrak{p}, \mathbb{V}}^{\tau}(\mathbf{x})$ satisfies the claim. This follows from Lemma 8.3 and Proposition 6.6. \square

9. AXIOMATIZATION OF $PS^\tau CC$ FIELDS

We use the results of the previous section to axiomatize the $PS^\tau CC$ property.

Definition 9.1. Construct an $\mathcal{L}_{\text{ring}}(K)$ -theory $T_{PS^\tau CC}$ as follows: Let

$$f_n(\mathbf{T}, \mathbf{Z}) = \sum_{\boldsymbol{\alpha}} T_{\boldsymbol{\alpha}} Z_1^{\alpha_1} \cdots Z_n^{\alpha_n} \in \mathbb{Z}[\mathbf{T}, \mathbf{Z}]$$

be the general polynomial in n variables Z_1, \dots, Z_n of degree n with coefficients \mathbf{T} . Here $\boldsymbol{\alpha}$ runs over all n -tuples $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{Z}_{\geq 0}$, $\sum_{i=1}^n \alpha_i \leq n$.

For $n \in \mathbb{N}$, let $\psi_n(\mathbf{x}, \mathbf{y})$ be an $\mathcal{L}_{\text{ring}}$ -formula stating that the polynomial $f_n(\mathbf{x}, \mathbf{Z})$ with coefficients \mathbf{x} is absolutely irreducible (see for example [FJ08, Chapter 11.3]), and all singular points on the affine hypersurface defined by this polynomial lie on the subvariety defined by the polynomial $f_n(\mathbf{y}, \mathbf{Z})$ with coefficients \mathbf{y} . Let $\eta_n(\mathbf{x}, \mathbf{y})$ be the $\mathcal{L}_{\text{ring}}$ -formula

$$(\exists \mathbf{z})(f_n(\mathbf{x}, \mathbf{z}) = 0 \wedge f_n(\mathbf{y}, \mathbf{z}) \neq 0)$$

stating that the polynomial with coefficients \mathbf{x} has a zero which is not a zero of the polynomial with coefficients \mathbf{y} . Let $(\hat{\eta}_n)_{\mathfrak{p}, \forall}^\tau(\mathbf{x}, \mathbf{y})$ be the $\mathcal{L}_{\text{ring}}(K)$ -formula that Proposition 8.4 attaches to η_n , and let φ_n be the $\mathcal{L}_{\text{ring}}(K)$ -sentence

$$(\forall \mathbf{x}, \mathbf{y})[(\psi_n(\mathbf{x}, \mathbf{y}) \wedge \bigwedge_{\mathfrak{p} \in S} (\hat{\eta}_n)_{\mathfrak{p}, \forall}^\tau(\mathbf{x}, \mathbf{y})) \rightarrow \eta_n(\mathbf{x}, \mathbf{y})].$$

Let $T_{PS^\tau CC}$ consist of the following sentences:

- (1) For every $\mathfrak{p} \in S$ and $\tau' \leq \tau$, the theory $T_{R, \mathfrak{p}}^{\tau'}$.
- (2) For every $n \in \mathbb{N}$, the sentence φ_n .

Lemma 9.2. *Let $\mathfrak{p} \in S$ and $F' \in CC_{\mathfrak{p}}^\tau(F)$, and let V be a smooth F -variety. Then $V(F') \neq \emptyset$ if and only if there exists $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^\tau(F'(V))$ with $\text{tp}(\mathfrak{P}) = \text{tp}(F')$.*

Proof. For $p_{\mathfrak{p}} \neq \infty$, this follows from [PR84, 7.8]; for $p_{\mathfrak{p}} = \infty$, it follows from [Pre84, 3.13]. \square

Proposition 9.3. *The field F satisfies $T_{PS^\tau CC}$ if and only if F is $PS^\tau CC$.*

Proof. First assume that F is $PS^\tau CC$. Then F is also $PS^\tau CL$ (cf. Remark 4.3) and hence satisfies (1) by Proposition 6.2 and Proposition 6.6. For all tuples \mathbf{a}, \mathbf{b} from F , if $F \models (\hat{\eta}_n)_{\mathfrak{p}, \forall}^\tau(\mathbf{a}, \mathbf{b})$ then $F_{\mathfrak{p}} \models \eta_n(\mathbf{a}, \mathbf{b})$ for every $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^\tau(F)$ by Proposition 4.9 and Proposition 8.4. Therefore, if $F \models \psi_n(\mathbf{a}, \mathbf{b}) \wedge \bigwedge_{\mathfrak{p} \in S} (\hat{\eta}_n)_{\mathfrak{p}, \forall}^\tau(\mathbf{a}, \mathbf{b})$, then the conditions

$$f_n(\mathbf{a}, \mathbf{Z}) = 0, \quad f_n(\mathbf{b}, \mathbf{Z}) \neq 0 \tag{9.1}$$

define a non-singular F -variety V which has an $F_{\mathfrak{p}}$ -rational point for every $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^\tau(F)$. Thus, since F is $PS^\tau CL$, V has an F -rational point, so $F \models \eta_n(\mathbf{a}, \mathbf{b})$. Consequently, F satisfies (2).

Conversely, assume that F satisfies $T_{PS^\tau CC}$. Let V be any smooth F -variety that has an F' -rational point for every $F' \in CC_S^\tau(F)$. Since $F'(V) = F'(V')$ for any open subvariety V' of V , Lemma 9.2 implies that the F' -rational points are Zariski-dense on V . So since V is birationally equivalent to a hypersurface, we can assume without loss of generality that V is given by tuples \mathbf{a} resp. \mathbf{b} from F as in (9.1). Thus, $F' \models \eta_n(\mathbf{a}, \mathbf{b})$ for every $F' \in CC_S^\tau(F)$, so $F \models (\hat{\eta}_n)_{\mathfrak{p}, \forall}^\tau(\mathbf{a}, \mathbf{b})$ by (1) and Proposition 8.4. Since F satisfies (2), $F \models \eta_n(\mathbf{a}, \mathbf{b})$, i.e. V has an F -rational point, and so F is $PS^\tau CC$. \square

Remark 9.4. Clearly, we just proved Theorem 1.1 of the introduction. Note that Proposition 9.3 gives an $\mathcal{L}_{\text{ring}}$ -axiomatization of PpC , PRC , and PC_M fields, cf. Remark 4.3. By Proposition 4.6 we also get an $\mathcal{L}_{\text{ring}}$ -axiomatization of the class of PCC fields in the Hilbert-type infinitary logic $L_{\omega_1, \omega}$. Note that the class of PCC fields is not elementary in our standard finitary logic $L_{\omega, \omega}$.

We can use our results to prove the conjecture posed in [Dar01, Remark 11]: Darnière calls a field F **RC-local** if it is PFC for $\mathcal{F} = \text{CC}(F)$, and **restricted RC-local** if every elementary extension of F satisfies the same property. Let \mathcal{F} be a finite family of fields taken among \mathbb{R} and the finite extensions of the fields $\hat{\mathbb{Q}}_p$, and denote by $\mathbf{Q}_{\mathcal{F}}$ the maximal Galois extension of \mathbb{Q} contained in every $F \in \mathcal{F}$. Then $\mathbf{Q}_{\mathcal{F}}$ is PCC by [MB89] and [GPR95]. Darnière conjectures that it is restricted RC-local and that $R_{\mathcal{F}}$, the intersection over all p -valuation rings, is $\mathcal{L}_{\text{ring}}$ -definable in $\mathbf{Q}_{\mathcal{F}}$. The first part of this conjecture follows from our axiomatization of PCC fields, the second part follows from Proposition 6.2.

Corollary 9.5. *Let $\mathfrak{p} \in S$ and let $\varphi(\mathbf{x})$ be an $\mathcal{L}_{\text{ring}}$ -formula. The $\mathcal{L}_{\text{ring}}(K)$ -formula $\hat{\varphi}_{\mathfrak{p}, \vee}^{\tau}(\mathbf{x})$ of Proposition 8.4 satisfies the following: For every $\text{PS}^{\tau}\text{CC}$ field $F \supseteq K$ and for all elements $a_1, \dots, a_m \in F$ the following are equivalent:*

- (1) $F \models \hat{\varphi}_{\mathfrak{p}, \vee}^{\tau}(\mathbf{a})$.
- (2) $F' \models \varphi(\mathbf{a})$ for all $F' \in \text{CC}_{\mathfrak{p}}^{\tau}(F)$.

Proof. This follows by combining Proposition 9.3, Proposition 4.9, and Proposition 8.4. \square

10. THE STRONG APPROXIMATION PROPERTY

We prove that the space of orderings of a $\text{PS}^{\tau}\text{CC}$ field satisfies the so called ‘strong approximation property’ of [Pre84], first studied in [KRW71]. We need the strong approximation property for the characterization of totally S^{τ} -adic extensions in terms of holomorphy domains, which follows in the next section.

Definition 10.1. Let $\tilde{\mathcal{S}}(F)$ be the set of *all* primes of F , and let $\tilde{\mathcal{S}}_{\mathfrak{p}}(F)$ be the subset of those lying above $\mathfrak{p} \in S$. We equip $\tilde{\mathcal{S}}(F)$ with the following **Zariski-topology**: A subbasis of open sets is given by sets of the form

$$H(a) = \{\mathfrak{P} \in \tilde{\mathcal{S}}(F) : a \in \mathcal{O}_{\mathfrak{P}}\},$$

where $a \in F$. A set $\mathcal{S} \subseteq \tilde{\mathcal{S}}(F)$ is **profinite** if \mathcal{S} , as a subspace of $\tilde{\mathcal{S}}(F)$, is a profinite space, i.e. a totally disconnected compact Hausdorff space. We say that \mathcal{S} satisfies **SAP** (the Strong Approximation Property) if \mathcal{S} is profinite and the family $H(a) \cap \mathcal{S}$, $a \in F$, is closed under finite intersections.

Let $\tilde{\mathcal{S}}_{\mathcal{P}}(F) = \tilde{\mathcal{S}}(F) \setminus \tilde{\mathcal{S}}_{\infty}(F)$ be the set of non-archimedean primes of F . We also consider the following (finer) **patch topology** on $\tilde{\mathcal{S}}_{\mathcal{P}}(F)$: A subbasis of open-closed sets is given by sets of the form

$$H_{\mathcal{P}}(a) = \{\mathfrak{P} \in \tilde{\mathcal{S}}_{\mathcal{P}}(F) : v_{\mathfrak{P}}(a) \geq 0\}$$

and

$$H'_{\mathcal{P}}(a) = \{\mathfrak{P} \in \tilde{\mathcal{S}}_{\mathcal{P}}(F) : v_{\mathfrak{P}}(a) > 0\},$$

where $a \in F$.

We say that F is S^{τ} -**SAP** if $\mathcal{S}_{\mathfrak{p}}^{\tau}(F)$ satisfies SAP for each $\mathfrak{p} \in S$.

Lemma 10.2. *Let $\mathfrak{p} \in S$ with $p_{\mathfrak{p}} \neq \infty$. The following subsets of $\tilde{\mathcal{S}}_{\mathfrak{p}}(F)$ are closed in the patch topology:*

- (1) $\mathcal{S}_{1,e'} := \{\mathfrak{P} \in \tilde{\mathcal{S}}_{\mathfrak{p}}(F) : v_{\mathfrak{P}} \text{ is discrete and } v_{\mathfrak{P}}(\pi_{\mathfrak{p}}) \leq e'\}, e' \in \mathbb{N}$
- (2) $\mathcal{S}_{2,f'} := \{\mathfrak{P} \in \tilde{\mathcal{S}}_{\mathfrak{p}}(F) : f' \nmid f_{\mathfrak{P}}\}, f' \in \mathbb{N}$
- (3) $\mathcal{S}_{\mathfrak{p}}^{\tau}(F)$

Proof. (1): For $\mathfrak{P} \in \tilde{\mathcal{S}}_{\mathfrak{p}}(F)$, $v_{\mathfrak{P}}(\pi_{\mathfrak{p}}) \leq e'$ if and only if for all $a \in F^{\times}$, $v_{\mathfrak{P}}(a) \leq 0$ or $v_{\mathfrak{P}}(a^{e'}) \geq v_{\mathfrak{P}}(\pi_{\mathfrak{p}})$, i.e.

$$\mathcal{S}_{1,e'} = \tilde{\mathcal{S}}_{\mathfrak{p}}(F) \cap \bigcap_{a \in F^{\times}} (H_{\mathcal{P}}(a^{-1}) \cup H_{\mathcal{P}}(\pi_{\mathfrak{p}}^{-1} a^{e'})).$$

(2): The following are equivalent: $f' \nmid f_{\mathfrak{P}}$; $\Phi_{p_{f'}-1}$ has a zero in $\bar{F}_{\mathfrak{P}}$; there exists $a \in F^{\times}$ with $v_{\mathfrak{P}}(a) \geq 0$ and $v_{\mathfrak{P}}(\Phi_{p_{f'}-1}(a)) > 0$. Thus,

$$\mathcal{S}_{2,f'} = \tilde{\mathcal{S}}_{\mathfrak{p}}(F) \cap \bigcap_{a \in F^{\times}} (H'_{\mathcal{P}}(a^{-1}) \cup H_{\mathcal{P}}(\Phi_{p_{f'}-1}(a)^{-1})).$$

(3): This follows from (1) and (2), since

$$\mathcal{S}_{\mathfrak{p}}^{\tau}(F) = \mathcal{S}_{1,e} \cap \bigcap_{f' \nmid f_{\mathfrak{p}}} \mathcal{S}_{2,f'}.$$

□

Lemma 10.3. *For every $\mathfrak{p} \in S$, $\mathcal{S}_{\mathfrak{p}}^{\tau}(F)$ is profinite, and the family $H(a) \cap \mathcal{S}_{\mathfrak{p}}^{\tau}(F)$, $a \in F$, is closed under complements (in $\mathcal{S}_{\mathfrak{p}}^{\tau}(F)$).*

Proof. First, assume that $p_{\mathfrak{p}} \neq \infty$. By [Kuh04, Corollary A.7], $\tilde{\mathcal{S}}_{\mathfrak{p}}(F)$ is quasi-compact and hence profinite in the patch topology. Thus, since $\mathcal{S}_{\mathfrak{p}}^{\tau}(F)$ is closed in $\tilde{\mathcal{S}}_{\mathfrak{p}}(F)$ by Lemma 10.2(3), also $\mathcal{S}_{\mathfrak{p}}^{\tau}(F)$ is profinite in the patch topology. Lemma 7.1(2) implies that the family $H_{\mathcal{P}}(a) \cap \mathcal{S}_{\mathfrak{p}}^{\tau}(F)$, $a \in F$, is closed under complements and the patch topology on $\mathcal{S}_{\mathfrak{p}}^{\tau}(F)$ coincides with the Zariski-topology, which proves the claim.

Now assume that $p_{\mathfrak{p}} = \infty$. Since for $a \in F^{\times}$, $\mathcal{S}_{\mathfrak{p}}(F) \setminus H(a) = H(-a) \cap \mathcal{S}_{\mathfrak{p}}(F)$, the family $H(a) \cap \mathcal{S}_{\mathfrak{p}}(F)$, $a \in F$, is closed under complements. By [Pre84, 6.5], the Zariski-topology on the space $\mathcal{S}_{\infty}(F)$ of orderings is profinite. Since $\mathcal{S}_{\mathfrak{p}}(F) = \bigcap_{a \in \mathcal{O}_{\mathfrak{p}}} H(a) \cap \mathcal{S}_{\infty}(F)$ is closed in $\mathcal{S}_{\infty}(F)$, the claim follows. □

Lemma 10.4. *If $p_{\mathfrak{p}} \neq \infty$, then $\mathcal{S}_{\mathfrak{p}}^{\tau}(F)$ satisfies SAP.*

Proof. This follows from Lemma 10.3 and Lemma 7.1(1). □

Lemma 10.5. *If F/K is algebraic, then F is S^{τ} -SAP.*

Proof. Let $\mathfrak{p} \in S$ and $a, b \in F$. Since F/K is algebraic, there exists a finite subextension L/K of F/K such that $a, b \in L$. The weak approximation theorem applied to the finite set of local primes (i.e. absolute values) $\mathcal{S}_{\mathfrak{p}}^{\tau}(L)$ yields $c \in L$ with $H(a) \cap H(b) \cap \mathcal{S}_{\mathfrak{p}}^{\tau}(F) = H(c) \cap \mathcal{S}_{\mathfrak{p}}^{\tau}(F)$. Thus, $\mathcal{S}_{\mathfrak{p}}^{\tau}(F)$ satisfies SAP. □

If F is PRC, then $\mathcal{S}_{\infty}(F)$ satisfies SAP, see [Pre81, Proposition 1.3]. In fact this holds for every $\text{PS}^{\tau}\text{CC}$ field. We prove this by combining the construction of Section 5 with the specific polynomial from [Pre81].

Lemma 10.6. *For $a, b \in F^{\times}$, let*

$$f_{a,b}(X, Y) = abX^2Y^2 + aX^2 + bY^2 - 1 \in F[X, Y].$$

If $\mathfrak{p} \in S$ with $p_{\mathfrak{p}} = \infty$, and $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}(F)$, then the following holds:

- (1) $f_{a,b}$ has a zero in $F_{\mathfrak{P}}$.

- (2) If $x, y \in F$ and $f_{a,b}(x, y) >_{\mathfrak{P}} -1$, then $ab(ax^2 + by^2) \geq_{\mathfrak{P}} 0$ if and only if $a \geq_{\mathfrak{P}} 0$ and $b \geq_{\mathfrak{P}} 0$.

Proof. (1): First note that

$$f_{a,b}(X, Y) = aX^2(bY^2 + 1) + (bY^2 - 1).$$

One can choose $y \in F$ such that $(-\frac{1}{a})\frac{by^2-1}{by^2+1} >_{\mathfrak{P}} 0$. Indeed, if $a >_{\mathfrak{P}} 0$, let $y = 0$. If $a <_{\mathfrak{P}} 0$ and $b >_{\mathfrak{P}} 0$, let $y = 1 + b^{-1}$. If $a <_{\mathfrak{P}} 0$ and $b <_{\mathfrak{P}} 0$, let $y = 1 - b^{-1}$. Since $F_{\mathfrak{P}}$ is real closed, there exists $x \in F_{\mathfrak{P}}$ such that $x^2 = (-\frac{1}{a})\frac{by^2-1}{by^2+1}$, hence $f_{a,b}(x, y) = 0$.

(2): First note that $f_{a,b}(0, 0) = -1$, so $x \neq 0$ or $y \neq 0$. Furthermore, $f_{a,b}(x, y) >_{\mathfrak{P}} -1$ implies that

$$ax^2 + by^2 >_{\mathfrak{P}} -abx^2y^2. \quad (10.1)$$

If $a >_{\mathfrak{P}} 0$ and $b >_{\mathfrak{P}} 0$, then $ab(ax^2 + by^2) \geq_{\mathfrak{P}} 0$. If $a <_{\mathfrak{P}} 0$ and $b <_{\mathfrak{P}} 0$, then $ab >_{\mathfrak{P}} 0$ and $ax^2 + by^2 <_{\mathfrak{P}} 0$ (since $x \neq 0$ or $y \neq 0$), so $ab(ax^2 + by^2) <_{\mathfrak{P}} 0$. If $a >_{\mathfrak{P}} 0$ and $b <_{\mathfrak{P}} 0$, or $a <_{\mathfrak{P}} 0$ and $b >_{\mathfrak{P}} 0$, then $ab <_{\mathfrak{P}} 0$ and thus $ab(ax^2 + by^2) <_{\mathfrak{P}} -a^2b^2x^2y^2 \leq_{\mathfrak{P}} 0$ by (10.1). \square

Proposition 10.7. *If F is $PS^{\tau}CC$, then F is S^{τ} -SAP.*

Proof. Let $\mathfrak{p} \in S$. If $p_{\mathfrak{p}} \neq \infty$, then $S_{\mathfrak{p}}^{\tau}(F)$ satisfies SAP by Lemma 10.4. Therefore, assume that $p_{\mathfrak{p}} = \infty$, and let $a, b \in F^{\times}$. We want to use the polynomials constructed in Section 5. Recall Lemma 6.1, which gives a translation from our current setting to Setting 5.1. In particular, write $S = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Assume $\mathfrak{p} = \mathfrak{p}_i$, and let

$$G_{a,b}(X, Y, Z) = H_2(Z)(1 - A_i(X)C(X)f_{a,b}(X, Y)) - H_2(1),$$

where A_i, C, H_u are the corresponding polynomials defined in Section 5, and $f_{a,b}$ is as in Lemma 10.6. By (A1), (C1), and Lemma 10.6(1), $A_i(X)C(X)f_{a,b}(X, Y)$ has a zero in $F_{\mathfrak{P}}$ for each $\mathfrak{P} \in \mathcal{S}_S^{\tau}(F)$. Since F is $PS^{\tau}CC$, (H3) and Lemma 5.4 imply that there exist $x, y, z \in F$ such that $G_{a,b}(x, y, z) = 0$. Thus, if $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}(F)$, then

$$1 - A_i(x)C(x)f_{a,b}(x, y) = \frac{H_2(1)}{H_2(z)} \leq_{\mathfrak{P}} \frac{5}{4},$$

by (H1) and (H2), so $A_i(x)C(x)f_{a,b}(x, y) \geq_{\mathfrak{P}} -1/4$. Since $A_i(x)C(x) >_{\mathfrak{P}} 1$ by (A5) and (C2), this implies that $f_{a,b}(x, y) \geq_{\mathfrak{P}} -1/4 >_{\mathfrak{P}} -1$. Therefore, by Lemma 10.6(2), $H(a) \cap H(b) \cap \mathcal{S}_{\mathfrak{p}}(F) = H(c) \cap \mathcal{S}_{\mathfrak{p}}(F)$, where $c = ab(ax^2 + by^2) \in F$. Hence, $\mathcal{S}_{\mathfrak{p}}(F)$ satisfies SAP, as claimed. \square

This proves Theorem 1.3 of the introduction. As Ido Efrat pointed out to me, there might be an alternative approach to Proposition 10.7 by deducing the SAP property from Galois theoretic properties of $PS^{\tau}CC$ fields, like in the real case in [Har90].

11. TOTALLY S^{τ} -ADIC FIELD EXTENSIONS

We conclude this work by defining totally S^{τ} -adic field extensions and describing them in terms of holomorphy domains. This also gives an equivalent definition of the $PS^{\tau}CC$ property.

Definition 11.1. Let $\mathfrak{p} \in S$ and $\tau' \leq \tau$. If M/F is an extension, let $\text{res}_{\mathfrak{p}}^{\tau'}: \mathcal{S}_{\mathfrak{p}}^{\tau'}(M) \rightarrow \mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$ be the restriction map given by $\mathfrak{Q} \mapsto \mathfrak{Q}|_F$. We call an extension M/F **totally S^{τ} -adic** if the restriction map $\text{res}_{\mathfrak{p}}^{\tau'}: \mathcal{S}_{\mathfrak{p}}^{\tau'}(M) \rightarrow \mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$ is surjective for each $\mathfrak{p} \in S$ and $\tau' \leq \tau$.

Remark 11.2. Note that the restriction map $\text{res}_{\mathfrak{p}}^{\tau'}: \mathcal{S}_{\mathfrak{p}}^{\tau'}(M) \rightarrow \mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$ is continuous in the Zariski-topology, and that M/F is totally S^{τ} -adic if and only if each $\mathfrak{P} \in \mathcal{S}_S^{\tau}(F)$ extends to a prime of M of the same type.

If $K = \mathbb{Q}$, $|S| = 1$, and $\tau = (1, 1)$ then our notion of totally S^{τ} -adic extensions coincides with the classical notions of totally real extensions (as in [Pre81], [Ers82]) resp. totally p -adic extensions (as in [Gro87], [Jar91]). The following lemmas unify results from these works.

Lemma 11.3. *The field F is $\text{PS}^{\tau}\text{CC}$ if and only if for every domain $R = F[x_1, \dots, x_n]$ which is finitely generated over F and whose quotient field M is regular and totally S^{τ} -adic over F , there exists an F -homomorphism $R \rightarrow F$.*

Proof. First assume that F is $\text{PS}^{\tau}\text{CC}$. If M/F is regular, then R is the coordinate ring of an affine F -variety V . Let $\mathfrak{p} \in S$ and $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^{\tau}(F)$. By Proposition 4.9, since F is $\text{PS}^{\tau}\text{CC}$, \mathfrak{P} is quasi-local. If M/F is totally S -adic, there exists $\mathfrak{Q} \in \mathcal{S}_{\mathfrak{p}}^{\tau}(M)$ with $\mathfrak{Q}|_F = \mathfrak{P}$ and $\text{tp}(\mathfrak{Q}) = \text{tp}(\mathfrak{P})$. Then $M_{\mathfrak{Q}} \supseteq F_{\mathfrak{P}}(V)$, so V has a smooth $F_{\mathfrak{P}}$ -rational point by Lemma 9.2. So since F is $\text{PS}^{\tau}\text{CC}$, V has an F -rational point, and therefore there exists an F -homomorphism $R \rightarrow F$.

Conversely, let V be a smooth F -variety that has an F' -rational point for every $F' \in \text{CC}_S^{\tau}(F)$. By Lemma 9.2 we can assume without loss of generality that V is affine. Then the coordinate ring $R = F[V]$ is a domain which is finitely generated over F and whose quotient field $M = F(V)$ is regular over F . Let $\mathfrak{p} \in S$, $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^{\tau}(F)$, and $F' \in \text{CC}(F, \mathfrak{P})$. There exists $\mathfrak{Q} \in \mathcal{S}_{\mathfrak{p}}^{\tau}(F'(V))$ with $\mathfrak{Q}|_F = \mathfrak{P}$ and $\text{tp}(\mathfrak{Q}) = \text{tp}(\mathfrak{P})$ by Lemma 9.2, so $\mathfrak{Q}|_M \in \mathcal{S}_{\mathfrak{p}}^{\tau}(M)$, $(\mathfrak{Q}|_M)|_F = \mathfrak{P}$, and $\text{tp}(\mathfrak{Q}|_M) = \text{tp}(\mathfrak{P})$. Hence, M/F is totally S^{τ} -adic, so by assumption there exists an F -homomorphism $R \rightarrow F$, i.e. V has an F -rational point, as claimed. \square

Lemma 11.4. *Let $\mathfrak{p} \in S$ and $\tau' \leq \tau$. If M/F is an extension and $\mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$ satisfies SAP, then the following statements are equivalent:*

- (1) $\text{res}_{\mathfrak{p}}^{\tau'}: \mathcal{S}_{\mathfrak{p}}^{\tau'}(M) \rightarrow \mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$ is surjective.
- (2) $R_{\mathfrak{p}}^{\tau'}(M) \cap F = R_{\mathfrak{p}}^{\tau'}(F)$.
- (3) $R_{\mathfrak{p}}^{\tau'}(M) \cap F \subseteq R_{\mathfrak{p}}^{\tau'}(F)$.

Proof. (1) \Rightarrow (2): Assume that $\text{res}_{\mathfrak{p}}^{\tau'}$ is surjective. Then

$$R_{\mathfrak{p}}^{\tau'}(F) = \bigcap_{\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}^{\tau'}(F)} \mathcal{O}_{\mathfrak{P}} = \bigcap_{\mathfrak{Q} \in \mathcal{S}_{\mathfrak{p}}^{\tau'}(M)} (\mathcal{O}_{\mathfrak{Q}} \cap F) = R_{\mathfrak{p}}^{\tau'}(M) \cap F.$$

(2) \Rightarrow (3): This is trivial.

(3) \Rightarrow (1): Assume that $\text{res}_{\mathfrak{p}}^{\tau'}$ is not surjective. By Lemma 10.3, $\mathcal{S}_{\mathfrak{p}}^{\tau'}(M)$ and $\mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$ are profinite spaces. Hence, since $\text{res}_{\mathfrak{p}}^{\tau'}$ is continuous, $\text{res}_{\mathfrak{p}}^{\tau'}(\mathcal{S}_{\mathfrak{p}}^{\tau'}(M))$ is closed in $\mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$. Therefore, $\mathcal{S}_{\mathfrak{p}}^{\tau'}(F) \setminus \text{res}_{\mathfrak{p}}^{\tau'}(\mathcal{S}_{\mathfrak{p}}^{\tau'}(M))$ is nonempty and open. It follows that the complement of a basic open-closed set contained in $\mathcal{S}_{\mathfrak{p}}^{\tau'}(F) \setminus \text{res}_{\mathfrak{p}}^{\tau'}(\mathcal{S}_{\mathfrak{p}}^{\tau'}(M))$ is an open-closed proper subset X of $\mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$ containing $\text{res}_{\mathfrak{p}}^{\tau'}(\mathcal{S}_{\mathfrak{p}}^{\tau'}(M))$. By Lemma 10.3, the subbasis $H(a) \cap \mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$, $a \in F$, of $\mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$ is closed under complements. Hence, since $\mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$ satisfies SAP, $X = H(x) \cap \mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$ for some $x \in F$ by [Pre84, 6.6]. Therefore,

$$\text{res}_{\mathfrak{p}}^{\tau'}(\mathcal{S}_{\mathfrak{p}}^{\tau'}(M)) \subseteq H(x) \cap \mathcal{S}_{\mathfrak{p}}^{\tau'}(F) \subsetneq \mathcal{S}_{\mathfrak{p}}^{\tau'}(F).$$

Then $x \in R_{\mathfrak{p}}^{\tau'}(M) \cap F$ but $x \notin R_{\mathfrak{p}}^{\tau'}(F)$, so $R_{\mathfrak{p}}^{\tau'}(M) \cap F \not\subseteq R_{\mathfrak{p}}^{\tau'}(F)$. \square

Corollary 11.5. *Assume that F is $PS^\tau CC$. If $F \prec M$ is an elementary extension, then M/F is regular and totally S^τ -adic.*

Proof. Every elementary extension is regular, see for example [FJ08, 7.3.3]. By Proposition 9.3, since F is $PS^\tau CC$ and $M \equiv F$, M is $PS^\tau CC$. Thus, by Proposition 6.2, since $F \prec M$, $R_{\mathfrak{p}}^{\tau'}(M) \cap F = R_{\mathfrak{p}}^{\tau'}(F)$ for each $\mathfrak{p} \in S$ and $\tau' \leq \tau$. By Proposition 10.7, since F is $PS^\tau CC$, F is S^τ -SAP, so $\mathcal{S}_{\mathfrak{p}}^{\tau'}(F)$ satisfies SAP for each $\mathfrak{p} \in S$ and $\tau' \leq \tau$. Therefore, by Lemma 11.4, M/F is totally S^τ -adic. \square

This finishes the proof of Corollary 1.4 of the introduction.

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